The Spectrum of Kleinian Manifolds

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1   Introduction

Let G be a real simple linear connected Lie group of real rank one, and let $\Gamma \subset G$ be a convex-cocompact, non-cocompact, torsion-free, discrete subgroup. This paper is devoted to the decomposition of the right regular representation of G on $L^2(\Gamma \backslash G)$ into irreducibles, the so called

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Plancherel decomposition. We also allow twists by finite-dimensional unitary representations (φ, V_φ) of Γ, i.e., we investigate the right regular representation of G on the space

$$L^2(\Gamma\backslash G, \varphi) := \{f : G \to V_\varphi \mid f(gx) = \varphi(g)f(x) \;\forall g \in \Gamma, \; x \in G, \; \int_{\Gamma\backslash G} |f(x)|^2 \, dx < \infty\} \; .$$

Let $K \subset G$ be a maximal compact subgroup. Then $X := G/K$ is a Riemannian symmetric space of negative curvature which is the universal covering of the locally symmetric space $Y := \Gamma\backslash X$. Our assumption on Γ implies that Y has infinite volume and no cusps. We call such a locally symmetric space a Kleinian manifold. Let (γ, V_γ) be a finite-dimensional unitary representation of K. Then we form the homogeneous vector bundle $V(\gamma) := G \times_K V_\gamma$ over X and the locally homogeneous vector bundle $V_Y(\gamma, \varphi) := \Gamma\backslash(V(\gamma) \otimes V_\varphi)$ over Y. Let $\mathfrak{g}$ denote the Lie algebra of G, $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{Z}$ its center. Via the left regular action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(X, V(\gamma))$ any $A \in \mathcal{Z}$ gives rise to a G-invariant differential operator. This operator descends to $C^\infty(Y, V_Y(\gamma, \varphi))$. Here the Casimir operator $\Omega_G \in \mathcal{Z}$, which defines an essentially selfadjoint elliptic operator of second order acting on $L^2(Y, V_Y(\gamma, \varphi))$, is of particular interest.

Our initial motivation was to obtain the spectral decomposition of the space of sections $L^2(Y, V_Y(\gamma, \varphi))$ of the bundle $V_Y(\gamma, \varphi)$ over Y with respect to the Casimir operator and other locally invariant differential operators. However, the isomorphism

$$L^2(Y, V_Y(\gamma, \varphi)) \cong [L^2(\Gamma\backslash G, \varphi) \otimes V_\gamma]^K$$

implies that the Plancherel decomposition of $L^2(\Gamma\backslash G, \varphi)$ is more or less equivalent to the desired spectral decompositions for all the bundles at once (for details see Section 11). Our main results are the Plancherel theorem Theorem 11.1 and its consequences for spectral decompositions obtained in Theorem 11.2.

The structure of the Plancherel decomposition depends on the critical exponent δ_Γ of Γ (see Definition 2.2). For technical reasons we have to exclude discrete subgroups of the isometry group of $X = \mathbb{O}H^2$ with $\delta_\Gamma \geq 0$ (because our method of the meromorphic continuation of Eisenstein series involves the "embedding trick" (see below) which does not work in this case). Then Theorem 11.1 provides a decomposition

$$L^2(\Gamma\backslash G, \varphi) = L^2(\Gamma\backslash G, \varphi)_{ac} \oplus L^2(\Gamma\backslash G, \varphi)_{cusp} \oplus L^2(\Gamma\backslash G, \varphi)_{\mathrm{res}} \oplus L^2(\Gamma\backslash G, \varphi)_U \; .$$

Here $L^2(\Gamma\backslash G, \varphi)_{ac}$ decomposes further into a sum of direct integrals corresponding to the unitary principal series representations of G, each occuring with infinite multiplicity, $L^2(\Gamma\backslash G, \varphi)_{cusp}$ decomposes into discrete series representations of G, each discrete series representation of G occurs with infinite multiplicity. These two parts, which are in a sense the main contribution to $L^2(\Gamma\backslash G, \varphi)$, have essentially the same structure as in the case of the trivial group, i.e., in the Plancherel theorem for $L^2(G)$ due to Okamoto [41], Hirai [30], and Harish-Chandra [25], [26]. The remaining two parts $L^2(\Gamma\backslash G, \varphi)_{\mathrm{res}}$ and $L^2(\Gamma\backslash G, \varphi)_U$ can only be non-trivial if $\delta_\Gamma \geq 0$. They consist of a direct sum of non-discrete series representations of G with real infinitesimal character occuring with finite multiplicity. The subsbscript res stands for residual spectrum. Indeed, the space $L^2(\Gamma\backslash G, \varphi)_{\mathrm{res}}$ is generated by residues of Eisenstein series. The "stable" subspace $L^2(\Gamma\backslash G, \varphi)_U$ is of similar nature but is orthogonal to the residues of Eisenstein series.

It contains representations of integral infinitesimal character only. The understanding of its significance deserves further study.

Concerning the spectral decomposition of the operator $-\Omega_G$ acting on $L^2(Y, V_Y(\gamma, \varphi))$ it follows that the absolute continuous spectrum consists of finitely many branches $[c_i, \infty]$ of infinite multiplicity, where the constants $c_i \in \mathbb{R}$ are computable in terms of γ, that there is no singular continuous spectrum, and that the discrete spectrum is finite. The set of eigenvalues with infinite multiplicity coincides with the discrete spectrum of $-\Omega_G$ on $L^2(X, V(\gamma))$. Eigenvalues of finite multiplicity can only occur if $\delta_\Gamma \geq 0$. The corresponding eigenspaces split into a residual and a "stable" part, where the residual part is generated by residues of Eisenstein series. The "stable" part can occur only if γ is non-trivial. Note that the Plancherel theorem also provides a finer decomposition of the generalized eigenspaces of the Casimir operator with respect to the algebra of locally invariant diffrential operators $D(G, \gamma)$. For instance, the Casimir operator may have embedded eigenvalues which are isolated with respect to $D(G, \gamma)$. For more information see Section 11.

Spectral decompositions of $L^2(Y, V_Y(\gamma, \varphi))$ (respectively partial results; sometimes also cusps are allowed) were previously obtained for

- trivial γ and surfaces by Patterson [44], compare also [23] and [20]

- higher dimensional real hyperbolic manifolds and trivial γ by Lax-Phillips [33], [34], [35], Perry [48], Mazzeo-Melrose [39], Mandouvalos [37]

- differential forms on real hyperbolic manifolds by Mazzeo-Phillips [40]

- differential forms on complex-hyperbolic manifolds by Epstein-Melrose-Mendoza [22], Epstein-Melrose [21].

As in most of these papers the crucial step towards a spectral decomposition is the construction and meromorphic continuation of Eisenstein series and the scattering matrix (at least up to the critical axis). Besides the papers just cited this problem for trivial γ and real hyperbolic manifolds Y is treated e.g. in [45], [46], [43], [16], [36], [1], [38], [49]. Our approach differs philosophically, if not technically, from these papers. After two sections of preparatory character Sections 4-7 contain the development of our geometric version of "scattering theory". The emphasis is on analysis on the sphere at infinity, i.e., the geodesic boundary ∂X of X. The advantage of this approach becomes manifest if one goes beyond the case of the trivial representation γ. Indeed, as experience shows, "meromorphic objects" (Eisenstein series, scattering matrices, Selberg zeta functions etc.) correspond to families of bundles on the boundary rather than to bundles on the (locally) symmetric space itself. The key notions which we are going to discuss here are the extension map ext and invariant distributions supported on the limit set. Very similar ideas appear in the work of van den Ban-Schlichtkrull and Delorme on the Plancherel formula for reductive symmetric spaces (see e.g. [5], [9], [14], [7]).

Let $P \subset G$ be a minimal parabolic subgroup. Then ∂X can be viewed as the homogeneous space G/P. Set $M := K \cap P$. Let $\sigma \in \hat{M}$ be an irreducible representation of M. Finite-

dimensional irreducible representations of P then come in families $\{\sigma_\lambda\}_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*}$ parametrized by a one-dimensional complex vector space $\mathfrak{a}_{\mathbb{C}}^*$. They determine families of homogeneous vector bundles $\{V(\sigma_\lambda)\}_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*}$ over ∂X. Sections of these bundles carry representations of G, the so-called principal series representations. We are interested in the space of Γ-invariant distribution sections $^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda))$ as well as in its twisted version $^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$, where φ is a finite-dimensional representation of Γ. It turns out to be useful to allow non-unitary twists φ, too. ∂X is the union of the domain of discontinuity Ω of Γ and the limit set Λ. The most interesting invariant distributions are those which are supported on the limit set. By a slight abuse of notation we denote the space of such distributions by $^{\Gamma}C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. A prominent example of an invariant distribution supported on the limit set is given by the Patterson-Sullivan measure which is an element in $^{\Gamma}C^{-\infty}(\Lambda, V(1_{\delta_\Gamma}))$. Here 1 stands for the trivial representation of M, and the critical exponent δ_Γ is viewed as an element of $\mathfrak{a}_{\mathbb{C}}^*$ in a natural way. Eventually, it will turn out that for unitary φ the representations appearing in $L^2(\Gamma \backslash G, \varphi)_{\mathrm{res}} \oplus L^2(\Gamma \backslash G, \varphi)_U$ can be parametrized by the set $\{(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda) \geq 0, {}^{\Gamma}C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \neq 0\}$.

In order to construct elements of $^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ we proceed as follows. We consider the boundary at infinity of Y, the compact manifold $B := \Gamma \backslash \Omega$. The bundle $V(\sigma_\lambda, \varphi)$ on ∂X induces a corresponding bundle $V_B(\sigma_\lambda, \varphi)$ on B such that

$$C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \cong {}^{\Gamma}C^{-\infty}(\Omega, V(\sigma_\lambda, \varphi)) \ .$$

Using this isomorphism we want to construct the extension map

$$ext : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to {}^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$$

which extends an invariant distribution on Ω across the limit set Λ. This is possible if λ belongs to some right half-plane depending on the critical exponent δ_Γ. In fact, we define ext to be the adjoint of an operator π_* which associates to each smooth section of $V(\tilde{\sigma}_{-\lambda}, \tilde{\varphi})$ the Γ-average of its restriction to Ω. Because of the convergence of the Poincare series this average exists for $\mathrm{Re}(\lambda)$ sufficiently large and depends holomorphically on λ in that region. The first task (Section 5) is to obtain a meromorphic continuation of ext to all of $\mathfrak{a}_{\mathbb{C}}^*$.

Classically, for trivial γ, Eisenstein series are obtained by averaging the Poisson kernel $P_\lambda(x, b)$, $x \in X$, $b \in \Omega \subset \partial X$, over Γ. Then the pairing of this Eisenstein series with a distribution ϕ on B yields the eigenfunction $E(\lambda, \phi)$ on Y, also called Eisenstein series. Using the extension map we can rewrite

$$E(\lambda, \phi) = P_\lambda \circ ext(\phi) \ , \tag{1}$$

where P_λ is the Poisson transform. It is Equation (1) which we will use to define Eisenstein series for general bundles. Thus, in our approach the extension map ext is the primary object. Once ext is understood, the Eisenstein series will not cause any essential additional difficulties.

There is a second important object closely related to ext, the scattering matrix S_λ which we define as follows

$$S_\lambda := res \circ J_\lambda \circ ext : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to C^{-\infty}(B, V_B(\sigma_{-\lambda}, \varphi)) \ , \tag{2}$$

where
$$res : {}^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$$

is induced by restriction from ∂X to Ω, and

$$J_\lambda : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(\partial X, V(\sigma_{-\lambda}, \varphi))$$

is the scattering matrix for the trivial group Γ known in representation theory as Knapp-Stein intertwining operator .

Initially, ext and S_λ are defined on some right half-plane. Their meromorphic continuation proceeds in three surprisingly simple steps which we are going to sketch now.

For step one assume that ext is defined on a half-plane $\{\mathrm{Re}(\lambda) > -\epsilon\}$ for some $\epsilon > 0$. Then we show the functional equations

$$\begin{align}
S_\lambda &= S_{-\lambda}^{-1} \tag{3}\\
ext &= J_{-\lambda} \circ ext \circ S_\lambda \tag{4}
\end{align}$$

for $|\mathrm{Re}(\lambda)| < \epsilon$. Under the additional hypotheses $\sigma = 1$ we use meromorphic Fredholm theory in order to show that $S_{-\lambda}^{-1}$ extends meromorphically to a much bigger half-plane. The main point here is that J_λ can be used to construct a nice family of parametrices for $S_{-\lambda}$. Now (3) and (4) give the continuation of S_λ and ext, respectively, to this half-plane.

The remaining two steps are purely algebraic in nature. In the second step we embed G, hence Γ, into the isometry group of a symmetric space of sufficiently large dimension. In this higher dimensional situation we can apply step one. By the first two steps we obtain the meromorphic continuation of ext and S_λ to a half-plane which is independent of δ_Γ. This "embeddding trick" has appeared already in [36] in the context of the meromorphic continuation of Eisenstein series. Unfortunately, it is not applicable to the exceptional case $X = \mathbb{O}H^2$.

In the third step we use tensoring with finite-dimensional G-representations in order to embed the bundle $V_B(\sigma_\lambda, \varphi)$ into a bundle of the form $V_B(1_\mu, \varphi')$ for a suitable representation φ' of Γ and $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ belonging to a region where ext is already known to be meromorphic. In this step it is crucial to allow non-unitary twists. Note that this method of meromorphic continuation is independent of any spectral theoretic argument.

In Section 6 we show how ext can be used to construct all invariant distributions $\phi \in {}^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$, in particular those supported on the limit set. Indeed, it follows from $res \circ ext = \mathrm{id}$ that at points $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, where ext has a pole, the leading singular part of its Laurent expansion at λ maps $C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ to ${}^{\Gamma}C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. This construction may be viewed as a generalization of the construction of the Patterson-Sullivan measure. The main result of this section is Theorem 6.1 stating the discreteness of the set of "resonances" $\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid {}^{\Gamma}C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \neq 0\}$ and that for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the space ${}^{\Gamma}C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ is finite-dimensional. In contrast to the meromorphic continuation of ext the proof of this theorem also requires analysis on X, in particular a detailed knowledge of asymptotics of Poisson transforms and a variant of Green's formula. Note that the asymptotic formulas we use are

simple consequences of asymptotic expansions of matrix coefficients known from representation theory. They are strong enough to imply also the asymptotics of Eisenstein series which will play a decisive role in the final proof of the Plancherel theorem. It also follows that the scattering matrix defined by (2) is really a scattering matrix in the sense that it determines the relation between the two leading asymptotic terms of the Eisenstein series. In Section 7 we return to the assumption that φ is unitary which allows us to gain more detailed information concerning the location of the singularities of ext. In particular, ext, and hence the Eisenstein series, are regular at non-zero imaginary λ.

Note that only a part of the results obtained up to Section 6 is really needed for the derivation of the Plancherel theorem. However, we believe that the present version of scattering theory also provides an adequate foundation for more ambitious tasks in analysis on Kleinian manifolds as there are trace formulas, the investigation of Selberg and Ruelle zeta functions or Paley-Wiener theorems. For instance, in [13] we have used this approach in the case of real hyperbolic manifolds in order to prove a conjecture of Patterson concerning the relation between invariant distributions supported on the limit set and the singularities of the (untwisted) Selberg zeta function. In a paper which is in preparation we extend this approach to scattering theory to geometrically finite groups Γ.

The spectral theoretic part of the paper starts with Section 8. The problem we have to solve is twofold: first to produce a certain amount of eigensections and wave packets of them, and second to show that they span the whole Hilbert space. While the first task is almost standard once the Eisenstein series are constructed the second requires additional arguments. In particular, one has to show the absence of the singular continuous spectrum. Usually, the limiting absorption principle (e.g. [48]) or commutator methods (see e.g. [24]) are employed at this point. Here we use a completely different method proposed by Bernstein [8]. This method, brought to our attention by Delorme's proof of the Plancherel theorem for reductive symmetric spaces [19], rests on a theory of appropriate Schwartz spaces for Y (or $\Gamma \backslash G$). It leads to the notion of tempered eigensections or, switching to representation theoretic language, tempered invariant distribution vectors of unitary representations of G. These notions will be discussed in Section 8. The crucial point is that a priori only tempered eigensections can occur in the spectral decomposition. In Section 9 we relate tempered invariant distribution vectors to invariant distributions on ∂X, in particular those supported on the limit set. Combining the results of Sections 6 and 7 with knowledge of the structure of the unitary dual $\hat{G}$ of G we obtain a classification of tempered invariant distribution vectors. This classification enables us to complete the exhaustion part of the proof of the Plancherel theorem in Section 11.

Some of the readers may have noticed an earlier version of this paper which appeared as an e-print more than two years ago. At that time we were not able to continue ext to all of $\mathfrak{a}_{\mathbb{C}}^*$, but only to the complement of a set of integer points. In particular, in order to conclude finiteness of the discrete spectrum we were forced to combine several not very natural arguments. Now the discreteness result Theorem 6.1 which is a consequence of the meromorphy of ext gives (among other things) a very satisfactory understanding of the finiteness of the discrete spectrum. In addition, the point of view in the present version is more representation theoretic. In our opinion, this makes the fine structure of the spectrum much more transparent.

2   Geometric preparations

Let G be a connected, linear, real simple Lie group of rank one, $G = KAN$ be an Iwasawa decomposition of G, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the corresponding Iwasawa decomposition of the Lie algebra $\mathfrak{g}$, $M := Z_K(A)$ be the centralizer of A in K, and $P := MAN$ be a minimal parabolic subgroup. The group G acts isometrically on the rank-one symmetric space $X := G/K$. Let $\partial X := G/P = K/M$ be its geodesic boundary. We consider $\bar{X} := X \cup \partial X$ as a compact manifold with boundary.

By the classification of symmetric spaces with strictly negative sectional curvature X is one of the following spaces:

- a real hyperbolic space $\mathbb{R}H^n$, $n \geq 1$,

- a complex hyperbolic space $\mathbb{C}H^n$, $n \geq 2$,

- a quaternionic hyperbolic space $\mathbb{H}H^n$, $n \geq 2$,

- or the Cayley hyperbolic plane $\mathbb{O}H^2$,

and G is a linear group finitely covering the orientation-preserving isometry group of X.

We consider a torsion-free discrete subgroup $\Gamma \subset G$ such that ∂X admits a Γ-invariant partition $\partial X = \Omega \cup \Lambda$, where $\Omega \neq \emptyset$ is open and Γ acts freely and cocompactly on $X \cup \Omega$. The closed subset Λ is called the limit set of Γ. The locally symmetric space $Y := \Gamma \backslash X$ is a complete Riemannian manifold of infinite volume without cusps. It can be compactified by adjoining the geodesic boundary $B := \Gamma \backslash \Omega$.

A subgroup Γ satisfying this assumption is often called convex-cocompact since it acts cocompactly on the convex hull of the limit set. The quotient Y can be called a Kleinian manifold generalizing the corresponding notion for three-dimensional hyperbolic manifolds.

By $\mathfrak{a}_{\mathbb{C}}^*$ we denote the comlexified dual of $\mathfrak{a}$. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, then we set $a^\lambda := e^{\langle \lambda, \log(a) \rangle} \in \mathbb{C}$. Let α be the short root of $\mathfrak{a}$ in $\mathfrak{n}$. We set $A_+ := \{a \in A \mid a^\alpha \geq 1\}$. Define $\rho \in \mathfrak{a}^*$ as usual by $\rho(H) := \frac{1}{2}\mathrm{tr}(\mathrm{ad}(H)_{|\mathfrak{n}})$, $\forall H \in \mathfrak{a}$. We have

X	$\mathbb{R}H^n$	$\mathbb{C}H^n$	$\mathbb{H}H^n$	$\mathbb{O}H^n$
ρ	$\frac{n-1}{2}\alpha$	$n\alpha$	$(2n+1)\alpha$	11α

.

We adopt the following conventions about the notation for points of X and ∂X. A point $x \in \partial X$ can equivalently be denoted by a subset $kM \subset K$ or $gP \subset G$ representing this point in $\partial X = K/M$ or $\partial X = G/P$. If $F \subset \partial X$, then $FM := \bigcup_{kM \in F} kM \subset K$. Analogously, we can denote $b \in X$ by $gK \subset G$, where gK represents b in $X = G/K$.

Lemma 2.1 *For any compact $F \subset \Omega$ we have $\sharp(\Gamma \cap (FM)A_+K) < \infty$.*

Proof. Note that $(FM)A_+K \cup F \subset X \cup \Omega$ is compact. Thus its intersection with the orbit ΓK of the origin of X is finite. $\qquad\square$

The geometry of the action of Γ on $X \cup \partial X$ can be studied in a uniform way using the various decompositions of G. Any element $g \in G$ has a Cartan decomposition $g = k_g a_g h$, $k_g, h \in K$, $a_g \in A_+$, where a_g and $k_g M \in K/M$ are uniquely determined by g. Let $g = \kappa(g)a(g)n(g)$, $\kappa(g) \in K$, $a(g) \in A$, $n(g) \in N$ be defined with respect to the given Iwasawa decomposition. The function $G \times K \ni (g,k) \mapsto a(g^{-1}k)$ descends to $X \times \partial X$.

Given a normalization of the invariant distance d on X we identify A_+ with $[1, \infty)$ such that $a = \mathrm{e}^{d(eK,aK)}$. Then for any $g \in G$ and $k \in K$ we have

$$a_g = \mathrm{e}^{d(eK,gK)} \quad \text{and} \quad a(g^{-1}k) = \mathrm{e}^{\pm d(eK,HS_{gK,kM})} \ , \tag{5}$$

where $HS_{gK,kM} = kNk^{-1}gK$ is the horosphere passing through $gK \in X$ and $kM \in \partial X$. The sign $\pm$ is positive (negative) if eK lies inside (outside) the corresponding horoball.

Definition 2.2 *The critical exponent $\delta_\Gamma \in \mathfrak{a}^*$ of Γ is the smallest element such that $\sum_{g \in \Gamma} a_g^{-(\lambda+\rho)}$ converges for all $\lambda \in \mathfrak{a}^*$ with $\lambda > \delta_\Gamma$. If Γ is the trivial group, then we set $\delta_\Gamma := -\infty$.*

Equation (5) shows that this definition of δ_Γ differs from the usual one by a ρ-shift, only.

The critical exponent δ_Γ has been extensively studied, in particular by Patterson [47], Sullivan [52], and Corlette [18]. From these papers we know that $\delta_\Gamma \in [-\rho, \rho)$, if Γ is non-trivial. Moreover, $\delta_\Gamma + \rho = \dim_H(\Lambda)\alpha$, where $\dim_H(\Lambda)$ denotes the Hausdorff dimension of the limit set with respect to the natural class of sub-Riemannian metrics on ∂X (the Hausdorff dimension of the empty set is by definition $-\infty$). If X is a quaternionic hyperbolic space or the Cayley hyperbolic plane, then δ_Γ can not be arbitrary close to ρ. In these cases we have $\delta_\Gamma \leq (2n-1)\alpha$ and $\delta_\Gamma \leq 5\alpha$, respectively [18].

We now collect some facts concerning the relation between the Cartan and the Iwasawa decomposition. First of all Equation (5) implies

$$a(g^{-1}k) \leq a_g \quad \text{for all } g \in G, \ k \in K \ . \tag{6}$$

Lemma 2.3 *Let $k_0 M \in \partial X$. For any compact $W \subset (\partial X \setminus k_0 M)M$ and any neighbourhood $U \subset K$ of $k_0 M$ satisfying $W \subset (\partial X \setminus \mathrm{clo}(U)M)M$ there exists a constant $C > 0$, such that*

$$C a_g \leq a(g^{-1}k) \leq a_g$$

for all $g = k_g a_g h \in KA_+K$ with $k_g \in W$, and all $k \in U$.

Proof. The upper bound is given by (6). We prove the lower bound. Let $w \in N_K(M)$ represent the non-trivial element of the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Set $\bar{N} = \theta(N)$, where θ is the Cartan involution of G fixing K. Since the set $W^{-1}\text{clo}(U)M$ is compact and disjoint from M, there is a precompact open $V \subset \bar{N}$ such that $W^{-1}\text{clo}(U)M \subset w\kappa(V)M$.

Let $k \in U$. Then we have $k_g^{-1}k = w\kappa(\bar{n})m$ for $\bar{n} \in V$, $m \in M$. Using that $a(\bar{n}) \geq 1$ for all $\bar{n} \in \bar{N}$ (see e.g. [28], Ch. IV, Cor. 6.6.) we obtain

$$
\begin{aligned}
a(g^{-1}k) &= a(a_g^{-1}k_g^{-1}k) \\
&= a(a_g^{-1}w\kappa(\bar{n})m) \\
&= a(a_g\kappa(\bar{n})) \\
&= a(a_g\bar{n}n(\bar{n})^{-1}a(\bar{n})^{-1}) \\
&= a(a_g\bar{n}a_g^{-1})a(\bar{n})^{-1}a_g \\
&\geq a(\bar{n})^{-1}a_g \ .
\end{aligned}
$$

Since V is precompact we have $C := \inf_{\bar{n} \in V} a(\bar{n})^{-1} > 0$. It follows that $a(g^{-1}k) \geq Ca_g$. $\qquad\square$

As a corollary we obtain a certain converse of the triangle inequality.

Corollary 2.4 *Let k_0, W, and U be as in Lemma 2.3. Then there exists a constant $C > 0$ such that for all $a \in A$, $k \in U$, and $g \in G$ with $k_g \in W$*

$$
a_{g^{-1}ka} \geq Ca_g a \ .
$$

Proof. Combining Equation (6) with Lemma 2.3 we obtain

$$
a_{g^{-1}ka} = a_{(g^{-1}ka)^{-1}} \geq a(g^{-1}ka) = a(g^{-1}k)a \geq Ca_g a \ .
$$

$\qquad\square$

A word concerning normalizations: The basic object will be a fixed invariant Riemannian metric on X. Here the reader has the freedom to choose his favoured one. The exponential map then induces a metric on $\mathfrak{a}$, hence on $\mathfrak{a}^*$. Throughout the paper we will often isometrically identify $\mathfrak{a}$ with $\mathbb{R}$, $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$, A_+ with $[1, \infty)$ such that (5) holds. Haar measures will be normalized as follows: The measures on A, $\mathfrak{a}^*$ are fixed by the above metric. Compact groups will always have total mass 1. The Haar measure of G is given by $dg = dk \, dx$, where dk is the Haar measure of K and dx is the Riemannian measure on $X = G/K$. Finally, we will normalize the Haar measure on $\bar{N}$ such that

$$
\int_{\bar{N}} a(\bar{n})^{-2\rho} \, d\bar{n} = 1 \ .
$$

3   Analytic preparations

Let (σ, V_σ) be a finite-dimensional unitary representation of M. For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we form the representation σ_λ of P on $V_{\sigma_\lambda} := V_\sigma$, which is given by $\sigma_\lambda(man) := \sigma(m)a^{\rho-\lambda}$. Let $V(\sigma_\lambda) := G \times_P V_{\sigma_\lambda}$ be the associated homogeneous bundle over $\partial X = G/P$. It induces a bundle on $B = \Gamma \backslash \Omega$ defined by $V_B(\sigma_\lambda) := \Gamma \backslash V(\sigma_\lambda)_{|\Omega}$.

Let $\tilde{\sigma}$ be the dual representation to σ. Then there are natural pairings

$$
\begin{aligned}
V(\tilde{\sigma}_{-\lambda}) \otimes V(\sigma_\lambda) &\rightarrow \Lambda^{max} T^* \partial X \\
V_B(\tilde{\sigma}_{-\lambda}) \otimes V_B(\sigma_\lambda) &\rightarrow \Lambda^{max} T^* B \ .
\end{aligned}
$$

The orientation of ∂X induces one of B. Employing these pairings and integration with respect to the fixed orientation we obtain identifications

$$
\begin{aligned}
C^{-\infty}(\partial X, V(\sigma_\lambda)) &= C^\infty(\partial X, V(\tilde{\sigma}_{-\lambda}))' \\
C^{-\infty}(B, V_B(\sigma_\lambda)) &= C^\infty(B, V_B(\tilde{\sigma}_{-\lambda}))' \ .
\end{aligned}
$$

As a K-homogeneous bundle we have a canonical identification $V(\sigma_\lambda) \cong K \times_M V_\sigma$. Thus $\bigcup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} V(\sigma_\lambda) \rightarrow \mathfrak{a}_\mathbb{C}^* \times \partial X$ has the structure of a trivial holomorphic family of bundles.

Let $\pi^{\sigma,\lambda}$ denote the representation of G on the space of sections of $V(\sigma_\lambda)$ given by the left-regular representation. Then $\pi^{\sigma,\lambda}$ is called a principal series representation of G. Note that there are different globalizations of this representation which are distinguished by the regularity of the sections (smooth, distribution etc.).

For any small open subset $U \subset B$ and diffeomorphic lift $\tilde{U} \subset \Omega$ the restriction $V_B(\sigma_\lambda)_{|U}$ is canonically isomorphic to $V(\sigma_\lambda)_{|\tilde{U}}$. Let $\{U_\alpha\}$ be a cover of B by open sets as above. Then

$$
\bigcup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} V_B(\sigma_\lambda) \rightarrow \mathfrak{a}_\mathbb{C}^* \times B
$$

can be given the structure of a holomorphic family of bundles by gluing the trivial families

$$
\bigcup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} V_B(\sigma_\lambda)_{|U} \cong \bigcup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} V(\sigma_\lambda)_{|\tilde{U}}
$$

together using the holomorphic families of gluing maps induced by $\pi^{\sigma,\lambda}(g)$, $g \in \Gamma$.

The structure of a holomorphic family of bundles allows us to consider holomorphic or smooth or continuous families of sections $\mathfrak{a}_\mathbb{C}^* \ni \mu \mapsto f_\mu \in C^{\pm\infty}(\partial X, V(\sigma_\mu))$, $\mathfrak{a}_\mathbb{C}^* \ni \mu \mapsto f_\mu \in C^{\pm\infty}(B, V_B(\sigma_\mu))$, respectively.

Let (φ, V_φ) be a finite-dimensional representation of Γ. We form the bundle $V(\sigma_\lambda, \varphi) := V(\sigma_\lambda) \otimes V_\varphi$ on ∂X carrying the tensor product action of Γ and define $V_B(\sigma_\lambda, \varphi) := \Gamma \backslash (V(\sigma_\lambda) \otimes$

$V_\varphi)_{|\Omega}$. Then we have the spaces of sections $C^{\pm\infty}(\partial X, V(\sigma_\lambda, \varphi))$ and $C^{\pm\infty}(B, V_B(\sigma_\lambda, \varphi))$ as well as the various notions of $\mathfrak{a}_\mathbb{C}^*$-parametrized families of sections.

When dealing with holomorphic families of vectors in topological vector spaces we will employ the following functional analytic facts. Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \ldots$ be complete locally convex topological vector spaces. A locally convex vector space is called a Montel space if its closed bounded subsets are compact. A Montel space is reflexive, i.e., the canonical map into its bidual is an isomorphism. Moreover, the dual space of a Montel space is again a Montel space.

Fact 3.1 *The space of smooth sections of a vector bundle and its topological dual are Montel spaces.*

We equip $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ with the topology of uniform convergence on bounded sets. Let $V \subset \mathbb{C}$ be open. A map $f : V \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is called holomorphic if for any $z_0 \in V$ there is a sequence $f_i \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ such that $f(z) = \sum_{n=0}^\infty f_i(z - z_0)^i$ converges for all z close to z_0. Let $f : V \setminus \{z_0\} \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ be holomorphic and $f(z) = \sum_{n=-N}^\infty f_i(z - z_0)^i$ for all $z \ne z_0$ close to z_0. Then we say that f is meromorphic and has a pole of order N at z_0. If f_i, $i = -N, \ldots, -1$, are finite dimensional, then f has, by definition, a finite-dimensional singularity. We call a subset $A \subset \mathcal{F} \times \mathcal{G}'$ sufficiently large if for $B \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ the condition $\langle \phi, B\psi \rangle = 0$, $\forall (\psi, \phi) \in A$, implies $B = 0$. Proofs of the following facts can be found in [13], Section 2.2.

Fact 3.2 *The following assertions are equivalent :*

1. *(i) $f : V \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is holomorphic.*

2. *(ii) f is continuous and there is a sufficiently large subset $A \subset \mathcal{F} \times \mathcal{G}'$ such that for all $(\psi, \phi) \in A$ the function $V \ni z \mapsto \langle \phi, f(z)\psi \rangle$ is holomorphic.*

Fact 3.3 *Let $f_i : V \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ be a sequence of holomorphic maps. Moreover let $f : V \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ be continuous such that for a sufficiently large subset $A \subset \mathcal{F} \times \mathcal{G}'$ the functions $\langle \phi, f_i\psi \rangle$, $(\psi, \phi) \in A$, converge locally uniformly in V to $\langle \phi, f\psi \rangle$. Then f is holomorphic, too.*

Fact 3.4 *Let $f : V \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ be continuous. Then the adjoint $f' : V \to \mathrm{Hom}(\mathcal{G}', \mathcal{F}')$ is continuous. If f is holomorphic, then so is f'.*

Fact 3.5 *Assume that $\mathcal{F}$ is a Montel space. Let $f : V \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and $f_1 : V \to \mathrm{Hom}(\mathcal{G}, \mathcal{H})$ be continuous. Then $f_1 \circ f : V \to \mathrm{Hom}(\mathcal{F}, \mathcal{H})$ is continuous. If f, f_1 are holomorphic, so is $f_1 \circ f$.*

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{F} \subset \mathcal{H}$ be a Fréchet space such that the embedding is continuous and compact. In the application we have in mind $\mathcal{H}$ will be some L^2- space of sections of a vector bundle over a compact closed manifold and $\mathcal{F}$ be the Fréchet space of smooth sections of this bundle. The continuous maps $\mathrm{Hom}(\mathcal{H}, \mathcal{F})$ will be called smoothing operators.

Let $V \subset \mathbb{C}$ be open and connected, and $V \ni z \to R(z) \in \mathrm{Hom}(\mathcal{H}, \mathcal{F})$ be a meromorphic family of smoothing operators with at most finite-dimensional singularities. Note that $R(z)$ is a meromorphic family of compact operators on $\mathcal{H}$ in a natural way.

Lemma 3.6 *If $1 - R(z)$ is invertible at some point $z \in V$ where $R(z)$ is regular, then*

$$(1 - R(z))^{-1} = 1 - S(z) \ ,$$

where $V \ni z \to S(z) \in \mathrm{Hom}(\mathcal{H}, \mathcal{F})$ is a meromorphic family of smoothing operators with at most finite-dimensional singularities.

Proof. We apply Reed-Simon IV [50], Theorem XIII.13 in order to conclude that $(1 - R(z))^{-1}$ is a meromorphic family of operators on $\mathcal{H}$ having at most finite-dimensional singularities. Making the ansatz $(1 - R(z))^{-1} = 1 - S(z)$, where apriori $S(z)$ is a meromorphic familiy of bounded operators on $\mathcal{H}$ with finite-dimensional singularities, we obtain $S = -R - R \circ S$. This shows that S is a meromorpic family in $\mathrm{Hom}(\mathcal{H}, \mathcal{F})$. $\square$

4 Push-down and extension

Distribution sections of $V_B(\sigma_\lambda, \varphi)$ can be identified with Γ-invariant sections of $V(\sigma_\lambda, \varphi)_{|\Omega}$. In order to extend these distributions across the limit set in an invariant way we will construct the extension map

$$ext : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \ .$$

We first introduce its "adjoint" which is the push-down

$$\pi_* : C^\infty(\partial X, V(\sigma_\lambda, \varphi)) \to C^\infty(B, V_B(\sigma_\lambda, \varphi)) \ .$$

Using the identification $C^\infty(B, V_B(\sigma_\lambda, \varphi)) = {}^\Gamma C^\infty(\Omega, V(\sigma_\lambda, \varphi))$ we define π_* by

$$\pi_*(f)(kM) = \sum_{g \in \Gamma} (\pi(g)f)(kM), \quad kM \in \Omega \ , \tag{7}$$

if the sum converges. Here $\pi(g)$ is the action induced by $\pi^{\sigma, \lambda}(g) \otimes \varphi(g)$.

Note that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a filtered algebra. Let $\mathcal{U}(\mathfrak{g})_m$, $m \in \mathbb{N}_0$, be the space of elements of degree less or equal than m. Choose some norm on V_φ. It induces a

norm $|.|$ on $V_\sigma \otimes V_\varphi$. For any m and bounded subset $A \subset \mathcal{U}(\mathfrak{g})_m$ we define the seminorm $\rho_{m,A}$ on $C^\infty(\partial X, V(\sigma_\lambda, \varphi))$ by

$$\rho_{m,A}(f) := \sup_{X \in A, k \in K} |f(\kappa(kX))| \ ,$$

where we consider f as a function on K with values in $V_\sigma \otimes V_\varphi$. These seminorms define the Fréchet topology of $C^\infty(\partial X, V(\sigma_\lambda, \varphi))$ (in fact a countable set of such seminorms is sufficient).

In order to describe the Fréchet topology on $C^\infty(B, V_B(\sigma_\lambda, \varphi))$ we fix an open cover $\{U_\alpha\}$ of B such that each U_α has a diffeomorphic lift $\tilde{U}_\alpha \subset \Omega$. Then we have canonical isomorphisms

$$C^\infty(\tilde{U}_\alpha, V(\sigma_\lambda, \varphi)) \cong C^\infty(U_\alpha, V_B(\sigma_\lambda, \varphi)) \ .$$

For any $U \in \{U_\alpha\}$ we define the topology of $C^\infty(\tilde{U}, V(\sigma_\lambda, \varphi))$ using the seminorms

$$\rho_{U,m,A}(f) := \sup_{X \in A, k \in \tilde{U}M} |f(\kappa(kX))| \ ,$$

where $m \in \mathbb{N}_0$ and $A \subset \mathcal{U}(\mathfrak{g})_m$ is bounded. Since $C^\infty(B, V_B(\sigma_\lambda, \varphi))$ maps to $C^\infty(U_\alpha, V_B(\sigma_\lambda, \varphi))$ by restriction for each α we obtain a system of seminorms defining the Fréchet topology of $C^\infty(B, V_B(\sigma_\lambda, \varphi))$.

Since Γ is finitely generated we can find an element $\mu \in \mathfrak{a}^*$, $\mu \geq 0$, and a constant C such that

$$||\varphi(g)|| \leq C a_g^\mu \quad \text{for all } g \in \Gamma \ . \tag{8}$$

Definition 4.1 *Let $\delta_\varphi \in \mathfrak{a}^*$ be the infimum of all $\mu \in \mathfrak{a}^*$ satisfying Equation (8) for some C. It is independent of the chosen norm. We call δ_φ the exponent of (φ, V_φ).*

Note that unitary representations have zero exponents. If $\tilde{\varphi}$ is the dual representation of φ, then $\delta_{\tilde{\varphi}} = \delta_\varphi$. Furthermore, if φ is the restriction of a finite-dimensional representation of G with highest $\mathfrak{a}$-weight μ, then $\delta_\varphi = \mu$.

Lemma 4.2 *If $\operatorname{Re}(\lambda) < -(\delta_\Gamma + \delta_\varphi)$, then the sum (7) converges for $f \in C^\infty(\partial X, V(\sigma_\lambda, \varphi))$ and defines a continuous map*

$$\pi_* : C^\infty(\partial X, V(\sigma_\lambda, \varphi)) \to C^\infty(B, V_B(\sigma_\lambda, \varphi)) \ .$$

Moreover, π_* depends holomorphically on λ.

Proof. Consider $U \in \{U_\alpha\}$. We want to estimate

$$C^\infty(\partial X, V(\sigma_\lambda, \varphi)) \ni f \mapsto res_{\tilde{U}} \circ \pi(g)f \in C^\infty(\tilde{U}, V(\sigma_\lambda, \varphi)) \ .$$

Let $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ be the coproduct and write $\Delta(X) = \sum_i X_i \otimes Y_i$. Fix $l \in \mathbb{N}_0$ and a bounded set $A \in \mathcal{U}(\mathfrak{g})_l$. Then for any $\epsilon > 0$ there is another bounded set $A_1 \subset \mathcal{U}(\mathfrak{g})_l$ depending on A such that

$$
\begin{aligned}
\rho_{U,l,A}(res_{\tilde{U}M} \circ \pi(g)f) &= \sup_{X \in A, k \in \tilde{U}} |(\pi(g)f)(\kappa(kX))| \\
&= \sup_{X \in A, k \in \tilde{U}M} | \sum_i a(g^{-1}\kappa(kX_i))^{\lambda - \rho}(\mathrm{id} \otimes \varphi(g))f(\kappa(g^{-1}\kappa(kY_i)))| \\
&\leq a_g^{\delta_\varphi + \epsilon} \sup_{X \in A_1, k \in \tilde{U}M} |a(g^{-1}kX)^{\lambda - \rho}| \sup_{X \in A_1, k \in \tilde{U}M} |f(\kappa(g^{-1}kX))| . \quad (9)
\end{aligned}
$$

The Poincaré-Birkhoff-Witt theorem gives a decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\bar{\mathfrak{n}})\mathcal{U}(\mathfrak{m})\mathcal{U}(\mathfrak{a}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}$. Let $q : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\bar{\mathfrak{n}})\mathcal{U}(\mathfrak{m})\mathcal{U}(\mathfrak{a})$ be the associated projection. Then for $g \in G$ and $X \in \mathcal{U}(\mathfrak{g})$ we have $\kappa(gX) = \kappa(gq(X))$, $a(gX) = a(gq(X))$.

Let $U_1 \subset \Omega$ be an open neighbourhood of $\tilde{U}$. Then by Lemma 2.1 the intersection $\Gamma \cap U_1MA_+K$ is finite. Let $W := (\partial X \setminus U_1)M$. As in the proof of Lemma 2.3 we can find a compact A_+-invariant set $V \subset \bar{\mathfrak{n}}$ such that $W^{-1}\tilde{U}M \subset w\kappa(V)M$. For $g = ha_gh' \in WA_+K$ and $k \in \tilde{U}M$ we obtain $h^{-1}k = w\kappa(\bar{n})m$ for some $\bar{n} \in V$, $m \in M$.

Let $X \in \mathcal{U}(\mathfrak{g})$. Then

$$
\begin{aligned}
\kappa(g^{-1}kX) &= \kappa(h'^{-1}a_g^{-1}h^{-1}kX) \\
&= h'^{-1}\kappa(a_g^{-1}w\kappa(\bar{n})mX) \\
&= h'^{-1}w\kappa(a_g\bar{n}n(\bar{n})^{-1}a(\bar{n})^{-1}mX) \\
&= h'^{-1}w\kappa(a_g\bar{n}a_g^{-1}a_g[n(\bar{n})^{-1}a(\bar{n})^{-1}mXm^{-1}a(\bar{n})n(\bar{n})]a_g^{-1})m \\
&= h'^{-1}w\kappa(a_g\bar{n}a_g^{-1}a_gq(n(\bar{n})^{-1}a(\bar{n})^{-1}mXm^{-1}a(\bar{n})n(\bar{n}))a_g^{-1})m .
\end{aligned}
$$

Since V is compact the sets $n(V)^{-1}a(V)^{-1}MA_1Ma(V)n(V) =: A_2 \subset \mathcal{U}(\mathfrak{g})_l$ and $q(A_2)$ are bounded. Conjugating $q(A_2)$ with A_+ gives clearly another bounded set $A_3 \subset \mathcal{U}(\mathfrak{g})_l$. We can find a bounded set $A_4 \subset \mathcal{U}(\mathfrak{g})_l$ such that $\kappa(a_g\bar{n}a_g^{-1}A_3) \subset \kappa(\kappa(a_g\bar{n}a_g^{-1})A_4)$ for all $a_g \in A_+$. This implies for $g \in WA_+K$ that

$$
\sup_{X \in A_1, k \in \tilde{U}M} |f(\kappa(g^{-1}kX))| \leq \rho_{l,A_4}(f) . \quad (10)
$$

We also have

$$
\begin{aligned}
a(g^{-1}kX) &= a(h'^{-1}a_g^{-1}h^{-1}kX) \\
&= a(a_g^{-1}w\kappa(\bar{n})mX) \\
&= a(a_g\kappa(\bar{n})mXm^{-1}) \\
&= a(a_g\bar{n}n(\bar{n})^{-1}a(\bar{n})^{-1}mXm^{-1}) \\
&= a(a_g\bar{n}a_g^{-1}a_gn(\bar{n})^{-1}a(\bar{n})^{-1}mXm^{-1}a(\bar{n})n(\bar{n})a_g^{-1})a(\bar{n})^{-1}a_g . \\
&= a(a_g\bar{n}a_g^{-1}a_gq(n(\bar{n})^{-1}a(\bar{n})^{-1}mXm^{-1}a(\bar{n})n(\bar{n}))a_g^{-1})a(\bar{n})^{-1}a_g .
\end{aligned}
$$

As in the proof of Lemma 2.3 there is a constant $C < \infty$ such that

$$|a(a_g\bar{n}a_g^{-1}a_g q(n(\bar{n})^{-1}a(\bar{n})^{-1}mXm^{-1}a(\bar{n})n(\bar{n}))a_g^{-1})^{\lambda-\rho}|a(\bar{n})^{\rho-\lambda} < C$$

for all $a_g \in A_+$, $\bar{n} \in V$, $m \in M$, and $X \in A_1$. It follows that

$$\sup_{X\in A_1, k\in\tilde{U}M} |a(g^{-1}kX)^{\lambda-\rho}| \leq Ca_g^{\lambda-\rho} \tag{11}$$

for almost all $g \in \Gamma$. The estimates (9), (10) and (11) together imply that the sum

$$C^l(\partial X, V(\sigma_\lambda, \varphi)) \ni f \mapsto \sum_{g\in\Gamma} res_{\tilde{U}} \circ \pi(g)f \in C^l(\tilde{U}, V(\sigma_\lambda, \varphi))$$

converges for $\mathrm{Re}(\lambda) < -(\delta_\Gamma + \delta_\varphi)$ and defines a continuous map of Banach spaces. This map depends holomorphically on λ by Fact 3.3.

Combining these considerations for all $U \in \{U_\alpha\}$ and $l \in \mathbb{N}_0$ we obtain that

$$\pi_* : C^\infty(\partial X, V(\sigma_\lambda, \varphi)) \to C^\infty(B, V_B(\sigma_\lambda, \varphi))$$

is defined and continuous for $\mathrm{Re}(\lambda) < -(\delta_\Gamma + \delta_\varphi)$. Moreover it is easy to see that π_* depends holomorphically on λ. $\qquad\square$

Definition 4.3 *For $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$ we define the extension map*

$$ext : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$$

to be the adjoint of

$$\pi_* : C^*(\partial X, V(\tilde{\sigma}_{-\lambda}, \tilde{\varphi})) \to C^*(B, V_B(\tilde{\sigma}_{-\lambda}, \tilde{\varphi})) \ .$$

This definition needs a justification. In fact, by Lemma 4.2 the extension exists, is continuous, and by Fact 3.4 it depends holomorphically on λ. Moreover, it is easy to see that the range of the adjoint of π_* consists of Γ-invariant vectors.

We now consider the left-inverse of ext, the restriction

$$res : {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \ .$$

The space ${}^\Gamma C^{-\infty}(\Omega, V(\sigma_\lambda, \varphi))$ of Γ-invariant vectors in $C^{-\infty}(\Omega, V(\sigma_\lambda, \varphi))$ can be canonically identified with the corresponding space $C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$. Composing this identification with the restriction $res_\Omega : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(\Omega, V(\sigma_\lambda, \varphi))$ we obtain the required restriction map res.

Lemma 4.4 *There exists a continuous map*

$$\widetilde{res} : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \ ,$$

which depends holomorphically on λ and coincides with res on ${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$.

Proof. Let $\tilde{\varphi}$ be the dual representation of φ. We exhibit $\widetilde{res}$ as the adjoint of a continuous map

$$\pi^* : C^\infty(B, V_B(\tilde{\sigma}_{-\lambda}, \tilde{\varphi})) \to C^\infty(\partial X, V(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))$$

which depends holomorphically on λ. Then the lemma follows from Fact 3.4.

Let $\{U_\alpha\}$ be a finite open cover of B such that each U_α has a diffeomorphic lift $\tilde{U}_\alpha \subset \Omega$. Choose a subordinated partition of unity $\{\chi_\alpha\}$. Pulling χ_α back to $\tilde{U}_\alpha$ and extending the resulting function by 0 we obtain a function $\tilde{\chi}_\alpha \in C^\infty(\partial X)$. We define

$$\pi^*(f) := \sum_\alpha \tilde{\chi}_\alpha f, \quad f \in C^\infty(B, V_B(\tilde{\sigma}_{-\lambda}, \tilde{\varphi})) \ ,$$

where we consider f as an element of $^\Gamma C^{-\infty}(\Omega, V(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))$. Then we set $\widetilde{res} := (\pi^*)'$.   □

Lemma 4.5 *We have* $res \circ ext = \mathrm{id}$.

Proof. Recall the definition of π^* from the proof of Lemma 4.4. Then $\pi_* \pi^*$ is the identity on $C^\infty(B, V_B(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))$. We obtain

$$res \circ ext = \widetilde{res} \circ ext = (\pi^*)' \circ (\pi_*)' = (\pi_* \pi^*)' = \mathrm{id} \ .$$

□

Let $C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ denote the space of distribution sections of $V(\sigma_\lambda, \varphi)$ with support in the limit set Λ. Since Λ is Γ-invariant $C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ is a subrepresentation of the representation $\pi^{\sigma,\lambda} \otimes \varphi$ of Γ on $C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. Note that $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ is exactly the kernel of res.

Lemma 4.6 *If* $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$ *and if ext is defined, then we have* $ext \circ res = \mathrm{id}$.

Proof. The assumption implies that res is injective. By Lemma 4.5 we have $res(ext \circ res - \mathrm{id}) = 0$.
□

In order to apply this lemma we have to check its assumption. In the course of the paper we will prove several vanishing results for $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. Our first will be

Theorem 4.7 *If* $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$, *then* $^\Gamma C^{-\omega}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$.

The proof of this theorem is the first of many instances in this paper where we employ the Poisson transform and its asymptotics.

Let γ be a finite-dimensional representation of K on V_γ and $T \in \text{Hom}_M(V_\sigma, V_\gamma)$. We will view sections of $V(\gamma)$ as functions from G to V_γ satisfying the usual K-invariance condition.

Definition 4.8 *The Poisson transform*

$$P := P_\lambda^T : C^{-\infty}(\partial X, V(\sigma_\lambda)) \to C^\infty(X, V(\gamma))$$

is defined by

$$(P\phi)(g) := \int_K a(g^{-1}k)^{-(\lambda+\rho)}\gamma(\kappa(g^{-1}k))T\phi(k)dk \ .$$

Here $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda))$ and the integral is a formal notation meaning that the distribution ϕ has to be applied to the smooth integral kernel.

The theory of the Poisson transform is well-known in the case $\sigma = \gamma = 1$ (see e.g. [51]). The general case has been worked out in [53], [57], [42]. The Poisson transform P intertwines the left-regular representations of G on $C^{-\infty}(\partial X, V(\sigma_\lambda))$ and $C^\infty(X, V(\gamma))$. If σ is irreducible, then the image of P consists of joint eigensections with respect to the action of the center $\mathcal{Z}$ of the universal enveloping algebra of $\mathfrak{g}$, where $\mathcal{Z}$ acts by the infinitesimal character of $\pi^{\sigma,\lambda}$. For any finite-dimensional representation (φ, V_φ) of Γ we denote the tensor product $V(\gamma) \otimes V_\varphi$ by $V(\gamma, \varphi)$. Then the transformation $P_\lambda^T \otimes \text{id}$, which we will also denote by P, intertwines the Γ-modules $C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ and $C^\infty(X, V(\gamma, \varphi))$.

For the proof of Theorem 4.7 we can assume that σ is irreducible. The asymptotic properties of the Poisson transform will be discussed in detail at the beginning of section 6. At this point we need the following two facts. We fix a minimal K-type γ of the principal series representation $C^\infty(\partial X, V(\sigma_\lambda))$ (see [31], Ch. XV for all that) and choose $0 \neq T \in \text{Hom}_M(V_\sigma, V_\gamma)$. Assume that $\text{Re}(\lambda) > 0$. Then for $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ and $f \in C^\infty(\partial X, V((\tilde{\gamma}_{|M})_{-\lambda}, \tilde{\varphi}))$ we have

$$\lim_{a \to \infty} a^{\rho-\lambda} \int_K \langle P\phi(ka), f(k) \rangle \ dk = c_\sigma(\lambda)\langle \phi, {}^tTf \rangle \ , \tag{12}$$

where $c_\sigma(\lambda) \neq 0$ (see Corollarry 6.3 below). In particular, P is injective. For any compact $U \subset K \setminus \text{supp}(\phi)M$ there are constants $\epsilon > 0$, $C < \infty$ such that for all $k \in U$ and $a \in A_+$

$$|P\phi(ka)| \leq Ca^{\text{Re}(\lambda)-\rho-\epsilon} \ . \tag{13}$$

This follows from Lemma 6.2.

We now prove Theorem 4.7 under the additional hypotheses $\text{Re}(\lambda) > 0$. Choose a minimal K-type γ and an injective $T \in \text{Hom}_M(V_\sigma, V_\gamma)$ as above. Let $\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ and $f \in C^\infty(\partial X, V((\tilde{\gamma}_{|M})_{-\lambda}, \tilde{\varphi}))$. We extend f to a section $\tilde{f} \in C^\infty(X, V(\tilde{\gamma}, \tilde{\varphi}))$ by $\tilde{f}(k_1ak_2) =$

$\chi(a)\tilde{\gamma}(k_2)^{-1}f(k_1)$, where $\chi \in C^\infty(A_+)$ is some cut-off function satisfying $\chi \equiv 0$ in a neighbourhood of $1 \in A_+$ and $\chi(a) = 1$ for $a \gg 1$. Let F be a fundamental domain of the action of Γ on X such that $\mathrm{clo}(F) \cap \partial X \subset \Omega$ is compact. There is a constant $c \in \mathbb{R}$ such that

$$\lim_{a\to\infty} a^{\rho-\lambda} \int_K \langle P\phi(ka), f(k)\rangle \, dk$$

$$= \quad c \lim_{a\to\infty} a^{-(\lambda+\rho)} \int_{\{x\in G \,|\, a\le a_x \le 2a\}} \langle P\phi(x), \tilde{f}(x)\rangle \, dx$$

$$= \quad c \lim_{a\to\infty} a^{-(\lambda+\rho)} \sum_{g\in\Gamma} \int_{\{x\in G \,|\, a\le a_{gx} \le 2a\}\cap FK} \langle(\mathrm{id}\otimes\varphi(g))P\phi(x), \tilde{f}(gx)\rangle \, dx \ .$$

Choose $\epsilon > 0$ such that the inequality (13) holds and $\epsilon' := \mathrm{Re}(\lambda) - (\delta_\Gamma + \delta_\varphi + 2\epsilon) > 0$. We now use Corollary 2.4 in order to estimate for $g \in \Gamma$

$$\left| a^{-(\lambda+\rho)} \int_{\{x\in G \,|\, a\le a_{gx}\le 2a\}\cap FK} \langle(\mathrm{id}\otimes\varphi(g))P\phi(x), \tilde{f}(gx)\rangle \, dx \right|$$

$$\le \quad C_0 \sup_{k\in K} |f(k)| \, a^{-(\mathrm{Re}(\lambda)+\rho)} \, a_g^{\delta_\varphi+\epsilon} \int_{\{x\in G \,|\, a\le a_g a_x\le C_1 a\}\cap FK} |P\phi(x)| \, dx$$

$$\le \quad C_2 \, a^{-(\mathrm{Re}(\lambda)+\rho)} \, a_g^{\delta_\varphi+\epsilon} \, (a_g^{-1}a)^{\mathrm{Re}(\lambda)+\rho-\epsilon}$$

$$= \quad C_2 \, a_g^{-(\delta_\Gamma+\rho+\epsilon')} \, a^{-\epsilon} \ .$$

The constants C_i, $i = 0, 1, 2$ are independent of $g \in \Gamma$. Since $\sum_{g\in\Gamma} a_g^{-(\delta_\Gamma+\rho+\epsilon')}$ converges we obtain

$$|a^{\rho-\lambda} \int_K \langle P\phi(ka), f(k)\rangle \, dk| \le C_3 \, a^{-\epsilon} \ .$$

It follows that

$$\lim_{a\to\infty} a^{\rho-\lambda} \int_K \langle P\phi(ka), f(k)\rangle \, dk = 0 \ .$$

Since f was arbitrary and ${}^tT$ is surjective we conclude from (12) that $\phi = 0$.

It remains to consider the case $\mathrm{Re}(\lambda) \le 0$. Let (π_μ, W_μ) be the irreducible finite-dimensional spherical representation (a representation which contains the trivial K-type) with highest weight $\mu \in \mathfrak{a}^*$. Then the highest $\mathfrak{a}$-weight space of W_μ is P-invariant, and it is isomorphic to $V_{1_{-(\mu+\rho)}}$ as a P-module. Hence we have an embedding of P-representations $V_{\sigma_\lambda} \hookrightarrow V_{\sigma_{\lambda+\mu}} \otimes W_{\mu\,|P}$. This leads to an embedding of the G-homogeneous bundles over ∂X

$$V(\sigma_\lambda) \hookrightarrow V(\sigma_{\lambda+\mu} \otimes \pi_\mu) \cong V(\sigma_{\lambda+\mu}) \otimes W_\mu$$

as well as of the corresponding spaces of sections. Note that G acts on $V(\sigma_{\lambda+\mu}) \otimes W_\mu$ with the tensor product action. Tensoring with V_φ we obtain an injective intertwining operator of Γ-modules

$$i_\mu : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \hookrightarrow C^{-\infty}(\partial X, V(\sigma_{\lambda+\mu}, \varphi\otimes\pi_\mu)) \ , \tag{14}$$

which is local. Thus, if $\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$, then $i_\mu(\phi) \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_{\lambda+\mu}, \varphi\otimes\pi_\mu))$. Moreover, $\mathrm{Re}(\lambda + \mu) > \delta_\Gamma + \delta_\varphi + \mu = \delta_\Gamma + \delta_{\varphi\otimes\pi_\mu}$. Now choose μ large enough such that $\mathrm{Re}(\lambda+\mu) > 0$. It follows from what we have shown above that $i_\mu(\phi) = 0$ and hence $\phi = 0$. This finishes the proof of Theorem 4.7.

5 Meromorphic continuation of ext

The extension ext forms a holomorphic family of maps depending on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (we have omitted this dependence in order to simplify the notation) which is defined now for $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$. In the present section we construct the meromorphic continuation of ext to all of $\mathfrak{a}_{\mathbb{C}}^*$ for $X \neq \mathbb{O}H^2$. Our main tool is the scattering matrix which we introduce below. The scattering matrix for the trivial group $\Gamma = \{1\}$ is the Knapp-Stein intertwining operator of the corresponding principal series representation. We first recall basic properties of the Knapp-Stein intertwining operators. Then we define the scattering matrix using the extension and the Knapp-Stein intertwining operators. We simultaneously obtain the meromorphic continuations of the scattering matrix and the extension map.

If σ is a representation of M, then its Weyl-conjugate σ^w, acting on the same vector space V_σ, is defined by $\sigma^w(m) := \sigma(w^{-1}mw)$, where $w \in N_K(\mathfrak{a})$ is a representative of the non-trivial element of the Weyl group $W(\mathfrak{g}, \mathfrak{a}) \cong \mathbb{Z}_2$. The Knapp-Stein intertwining operators form meromorphic families of G-equivariant operators (see [32] and Lemmas 5.1 and 5.2 below)

$$\hat{J}_{\sigma,\lambda}^w : C^*(\partial X, V(\sigma_\lambda)) \to C^*(\partial X, V(\sigma_{-\lambda}^w)), \quad * = -\infty, \infty .$$

Here $\hat{\ }$ indicates that $\hat{J}_{\sigma,\lambda}^w$ is unnormalized.

In order to fix our conventions we give a definition of $\hat{J}_{\sigma,\lambda}^w$ as an integral operator acting on smooth functions for $\mathrm{Re}(\lambda) < 0$. Its continuous extension to distributions is obtained by duality. For $\mathrm{Re}(\lambda) \geq 0$ it is defined by meromorphic continuation.

Consider $f \in C^\infty(\partial X, V(\sigma_\lambda))$ as a function on G with values in V_{σ_λ} satisfying $f(gp) = \sigma_\lambda(p)^{-1} f(g)$ for all $p \in P$. For $\mathrm{Re}(\lambda) < 0$ the intertwining operator is defined by the convergent integral

$$(\hat{J}_{\sigma,\lambda}^w f)(g) := \int_{\bar{N}} f(gw\bar{n}) d\bar{n} . \tag{15}$$

Our first goal is to show that the intertwining operators form a meromorphic family of operators in the sense defined in Section 3. It is an easy consequence of the approach to the intertwining operators developed by Vogan-Wallach (see [55], Ch. 10). The only additional point we have to verify is that in the domain of convergence of (15) the operators $\hat{J}_{\sigma,\lambda}^w$ indeed form a continuous family.

Lemma 5.1 *For* $\mathrm{Re}(\lambda) < 0$ *the intertwining operators*

$$\hat{J}_{\sigma,\lambda}^w : C^\infty(\partial X, V(\sigma_\lambda)) \to C^\infty(\partial X, V(\sigma_{-\lambda}^w))$$

form a holomorphic family of continuous operators.

Proof. Let X_i, $i = 1, \ldots, \dim(\mathfrak{k})$, be an orthonormal base of $\mathfrak{k}$. For any multiindex $r = (i_1, \ldots, i_{\dim(\mathfrak{k})})$ we set $X_r = \prod_{l=1}^{\dim(\mathfrak{k})} X_l^{i_l}$, $|r| = \sum_{l=1}^{\dim(\mathfrak{k})} i_l$, and for $f \in C^\infty(K, V_{\sigma_\lambda})$ we define the

seminorm

$$\|f\|_r = \sup_{k \in K} |f(X_r k)| \ .$$

It is well known that the system $\{\|.\|_r\}$, r running over all multiindices, defines the Fréchet topology of $C^\infty(K, V_{\sigma_\lambda})$ and by restriction the topology of $C^\infty(\partial X, V(\sigma_\lambda))$.

We extend $f \in C^\infty(K, V_{\sigma_\lambda})$ to a function f_λ on G by setting $f_\lambda(kan) := f(k)a^{\lambda - \rho}$. Then we have

$$\hat{J}^w_{\sigma,\lambda}(f)(k) = \int_{\bar{N}} f_\lambda(kw\bar{n}) d\bar{n} \ .$$

For any $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$ with $\mathrm{Re}(\lambda) < 0$ and $\delta > 0$ we can find an $\epsilon > 0$ such that for $|\lambda - \lambda_0| < \epsilon$

$$\int_{\bar{N}} |a(\bar{n})^{\lambda_0 - \rho} - a(\bar{n})^{\lambda - \rho}| d\bar{n} < \delta \ .$$

We obtain

$$
\begin{aligned}
\|\hat{J}^w_{\sigma,\lambda_0} f - \hat{J}^w_{\sigma,\lambda} f\|_r &= \sup_{k \in K} \int_{\bar{N}} (f_{\lambda_0}(X_r kw\bar{n}) - (f_\lambda(X_r kw\bar{n})) d\bar{n} \\
&= \sup_{k \in K} \int_{\bar{N}} f(X_r kw\kappa(\bar{n}))(a(\bar{n})^{\lambda_0 - \rho} - a(\bar{n})^{\lambda - \rho}) d\bar{n} \\
&\leq \|f\|_r \int_{\bar{N}} |a(\bar{n})^{\lambda_0 - \rho} - a(\bar{n})^{\lambda - \rho}| d\bar{n} \\
&\leq \delta \|f\|_r
\end{aligned}
$$

This immediately implies that $\lambda \mapsto \hat{J}^w_{\sigma,\lambda}$ is a continuous family of operators on the space of smooth functions. The fact that the family $\hat{J}^w_{\sigma,\lambda}$, $\mathrm{Re}(\lambda) < 0$, depends holomorphically on λ is now easy to check (apply [55], Lemma 10.1.3 and Fact 3.2). $\qquad\square$

Lemma 5.2 *The family of intertwining operators*

$$\hat{J}^w_{\sigma,\lambda} : C^\infty(\partial X, V(\sigma_\lambda)) \to C^\infty(\partial X, V(\sigma^w_{-\lambda}))$$

extends meromorphically to all of $\mathfrak{a}^*_{\mathbb{C}}$.

Proof. We employ [55], Thm. 10.1.5, which states that there are polynomial maps $b : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ and $D : \mathfrak{a}^*_{\mathbb{C}} \to \mathcal{U}(\mathfrak{g})^K$, such that

$$b(\lambda) \hat{J}^w_{\sigma,\lambda} = \hat{J}^w_{\sigma,\lambda - 4\rho} \circ \pi^{\sigma, \lambda - 4\rho}(D(\lambda)) \ . \tag{16}$$

This formula requires some explanation. We identify

$$C^\infty(\partial X, V(\sigma_\lambda)) \cong C^\infty(K, V_\sigma)^M$$

canonically. Then all operators act on the same space $C^\infty(K, V_\sigma)^M$.

If we know that $\hat{J}_{\sigma,\lambda}^w$ is meromorphic up to $\mathrm{Re}(\lambda) < \mu$, then we conclude that

$$\hat{J}_{\sigma,\lambda}^w = \frac{1}{b(\lambda)} \hat{J}_{\sigma,\lambda-4\rho}^w \circ \pi^{\sigma,\lambda-4\rho}(D(\lambda))$$

is meromorphic up to $\mathrm{Re}(\lambda) < \mu + 4\rho$. Thus the lemma follows from Lemma 5.1. $\qquad\square$

Lemma 5.3 *Let* $\chi, \phi \in C^\infty(\partial X)$ *such that* $\mathrm{supp}(\phi) \cap \mathrm{supp}(\chi) = \emptyset$. *Then* $\chi \hat{J}_{\sigma,\lambda}^w \phi$ *is a holomorphic family of smoothing operators. In particular, the residues of* $\hat{J}_{\sigma,\lambda}^w$ *are differential operators.*

Proof. Since $\mathrm{supp}(\phi) \cap \mathrm{supp}(\chi) = \emptyset$, there exists a compact set $V \subset \bar{N}$ such that

$$\kappa(\mathrm{supp}(\chi) M w(\bar{N} \setminus V)) M \subset (\partial X \setminus \mathrm{supp}(\phi)) M \ .$$

For $\mathrm{Re}(\lambda) < 0$ and $f \in C^\infty(\partial X, V(\sigma_\lambda))$ we have (viewing f as a function on K with values in V_{σ_λ})

$$
\begin{aligned}
(\chi \hat{J}_{\sigma,\lambda}^w \phi f)(k) &= \int_{\bar{N}} \chi(k) f(\kappa(kw\bar{n})) \phi(\kappa(kw\bar{n})) a(\bar{n})^{\lambda-\rho} d\bar{n} \\
&= \int_V \chi(k) f(\kappa(kw\bar{n})) \phi(\kappa(kw\bar{n})) a(\bar{n})^{\lambda-\rho} d\bar{n} \ .
\end{aligned}
$$

The right-hand side of this equation extends to all of $\mathfrak{a}_{\mathbb{C}}^*$ and defines a holomorphic family of operators. This proves the first part of the lemma. It in particular implies that the residues of $\hat{J}_{\sigma,\lambda}^w$ are local operators. Hence the second assertion follows. $\qquad\square$

We need the following consequence of Lemma 5.3. Let $W \subset \partial X$ be a closed subset and let

$$\mathcal{G}_\lambda := \{f \in C^{-\infty}(\partial X, V(\sigma_\lambda)) | f_{|\partial X \setminus W} \in C^\infty(\partial X \setminus W, V(\sigma_\lambda))\} \ .$$

We equip $\mathcal{G}_\lambda$ with the weakest topology such that the maps $\mathcal{G}_\lambda \hookrightarrow C^{-\infty}(\partial X, V(\sigma_\lambda))$ and $\mathcal{G}_\lambda \to C^\infty(\partial X \setminus W, V(\sigma_\lambda))$ are continuous. Let $U \subset \bar{U} \subset \partial X \setminus W$ be open.

Corollary 5.4 *The composition*

$$res_U \circ \hat{J}_{\sigma,\lambda}^w : \mathcal{G}_\lambda \to C^\infty(U, V(\sigma_{-\lambda}^w))$$

is well-defined and depends meromorphically on λ. $\qquad\square$

Below we shall work with a slight modification of $\hat{J}_{\sigma,\lambda}^w$ not depending on the particular representative w and having the intertwininig properties we wish.

If σ is equivalent to σ^w, then we say that σ is Weyl-invariant. Unless indicated otherwise σ shall always denote a Weyl-invariant representation of M which is either irreducible or of the form $\sigma' \oplus \sigma'^w$ with σ' irreducible and not Weyl-invariant. In both cases the representation of M on V_σ can be extended to a representation of $N_K(\mathfrak{a})$ which we also denote by σ. This extension is unique up to a character of the Weyl group, i.e., the two possible choices of $\sigma(w)$ can differ by a sign, only. We fix such an extension and define the G-intertwining operator

$$\hat{J}_{\sigma,\lambda} : C^{\pm\infty}(\partial X, V(\sigma_\lambda)) \to C^{\pm\infty}(\partial X, V(\sigma_{-\lambda}))$$

by $\hat{J}_{\sigma,\lambda} := \sigma(w)\hat{J}^w_{\sigma,\lambda}$. Then the operator $\hat{J}_{\sigma,\lambda}$ does not depend on the choice of w.

In order to define normalized intertwining operators we first have to recall properties of c-functions and the functional equation of $\hat{J}_{\sigma,\lambda}$. Let γ be a finite-dimensional representation of K and let $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$. We define $T^w \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ by

$$T^w := \gamma(w)T\sigma(w^{-1}) \ .$$

T^w does not depend on the choice of the representative w. In a similar manner for $T \in \mathrm{End}_M(V_\gamma)$ we define $T^w := \gamma(w)T\gamma(w^{-1})$. Let $c_\gamma : \mathfrak{a}^*_{\mathbb{C}} \to \mathrm{End}_M(V_\gamma)$ be the meromorphic function given for $\mathrm{Re}(\lambda) > 0$ by

$$c_\gamma(\lambda) := \int_{\bar{N}} a(\bar{n})^{-(\lambda+\rho)}\gamma(\kappa(\bar{n})) \, d\bar{n} \ . \tag{17}$$

Lemma 5.5 *Let $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ and define $T^\sharp \in \mathrm{Hom}_K(C^{-\infty}(\partial X, V(\sigma_\lambda)), V_\gamma)$ by $T^\sharp(f) := (P^T_\lambda f)(1)$, $f \in C^{-\infty}(\partial X, V(\sigma_\lambda))$.*

1. *We have $T^\sharp \circ \hat{J}_{\sigma,\lambda} = [(c_\gamma(\lambda)T)^w]^\sharp$.*

2. *The Poisson transform satisfies the following functional equation*

$$P^T_\lambda \circ \hat{J}_{-\lambda} = P^{(c_\gamma(\lambda)T)^w}_{-\lambda} \ . \tag{18}$$

3. *There is a meromorphic function $p_\sigma : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ such that*

$$\hat{J}_{\sigma,-\lambda} \circ \hat{J}_{\sigma,\lambda} = \frac{1}{p_\sigma(\lambda)}\mathrm{id} \ . \tag{19}$$

4. *We have*

$$c_\gamma(-\lambda)^w c_\gamma(\lambda)T = \frac{1}{p_\sigma(\lambda)}T \ . \tag{20}$$

5. *The restriction of $\hat{J}_{\tilde{\sigma},\lambda}$ to $C^\infty(\partial X, V(\tilde{\sigma}_\lambda))$ coincides with the adjoint of $\hat{J}_{\sigma,-\lambda}$.*

6. *We have*

$$c_\gamma(\lambda)^* = c_\gamma(\bar{\lambda})^w \ . \tag{21}$$

Proof. The identity 1. immediately follows from (18). The latter can be read off from the asymptotics of the Poisson transforms (38) and (39). The functional equation

$$\hat{J}_{\sigma^w,-\lambda}^{w^{-1}} \circ \hat{J}_{\sigma,\lambda}^w = \frac{1}{p_\sigma(\lambda)} \mathrm{id}$$

can be found in [32]. We compute

$$
\begin{aligned}
\hat{J}_{\sigma,-\lambda} \circ \hat{J}_{\sigma,\lambda} &= \sigma(w) \circ \hat{J}_{\sigma,-\lambda}^w \circ \sigma(w) \circ \hat{J}_{\sigma,\lambda}^w \\
&= \sigma(w^{-1}) \circ \hat{J}_{\sigma,-\lambda}^{w^{-1}} \circ \sigma(w) \circ \hat{J}_{\sigma,\lambda}^w \\
&= \hat{J}_{\sigma^w,-\lambda}^{w^{-1}} \circ \sigma(w^{-1}) \circ \sigma(w) \circ \hat{J}_{\sigma,\lambda}^w \\
&= \hat{J}_{\sigma^w,-\lambda}^{w^{-1}} \circ \hat{J}_{\sigma,\lambda}^w \\
&= \frac{1}{p_\sigma(\lambda)} \mathrm{id} \ .
\end{aligned}
$$

Equation (20) can be obtained combining (19) with (18). The relation 5. follows from [32], Lemma 24. and (21) is a consequence of 5. and (20). □

Explicit formulas for the Plancherel density p_σ can be found e.g. in [32], Ch. 12.

For any (irreducible) $\sigma \in \hat{M}$ we fix a minimal K-type $\gamma_\sigma \in \hat{K}$ of $\pi^{\sigma,\lambda}$. Note that $\mathrm{Hom}_M(V_\sigma, V_{\gamma_\sigma})$ is one-dimensional. Therefore we can define a meromorphic function $c_\sigma : \mathfrak{a}_\mathbb{C}^* \mapsto \mathbb{C}$ such that

$$c_{\gamma_\sigma}(\lambda)T = c_\sigma(\lambda) \begin{cases} T^w, & \sigma = \sigma^w \\ T, & \sigma \neq \sigma^w \end{cases} \ . \tag{22}$$

If σ is of the form $\sigma' \oplus \sigma'^w$ for σ' not Weyl-invariant, then we define $c_\sigma(\lambda) \in \mathrm{End}_M(V_\sigma)$ such that it acts on the M-isotypic components $V_\sigma(\sigma')$ $(V_\sigma(\sigma'^w))$ as multiplication by $c_{\sigma'}(\lambda)$ $(c_{\sigma'^w}(\lambda))$.

Let now σ be a Weyl-invariant representation of M satisfying our general convention introduced above. We define the normalized intertwining operator by

$$J_{\sigma,\lambda} := \hat{J}_{\sigma,\lambda} c_\sigma(-\lambda)^{-1} \ .$$

Combining (19) and (20) we obtain the following functional equation:

$$J_{\sigma,-\lambda} \circ J_{\sigma,\lambda} = \mathrm{id} \ . \tag{23}$$

We shall often omit the subscript σ in the notation of the intertwining operators. Tensoring with a finite-dimensional representation (φ, V_φ) of Γ we obtain a meromorphic-family of Γ-intertwining operators which we denote by the same symbol

$$\hat{J}_\lambda := \hat{J}_{\sigma,\lambda} \otimes \mathrm{id} : C^{\pm\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{\pm\infty}(\partial X, V(\sigma_{-\lambda}, \varphi))$$

and its normalized version J_λ. Sometimes we are forced to consider also intertwining operators for irreducible, non-Weyl-invariant representations σ. Then $\hat{J}_{\sigma,\lambda}$ denotes the restriction of $\hat{J}_{\sigma \oplus \sigma^w,\lambda}$ to $C^{\pm\infty}(\partial X, V(\sigma_\lambda, \varphi))$.

We now turn to the definition of the (normalized) scattering matrix as a family of operators

$$\hat{S}_\lambda \quad : \quad C^*(B, V_B(\sigma_\lambda, \varphi)) \to C^*(B, V_B(\sigma_{-\lambda}, \varphi)), \quad * = \infty, -\infty$$
$$S_\lambda \quad : \quad C^*(B, V_B(\sigma_\lambda, \varphi)) \to C^*(B, V_B(\sigma_{-\lambda}, \varphi)) \ .$$

Definition 5.6 *For* $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$ *we define*

$$\hat{S}_\lambda := res \circ \hat{J}_\lambda \circ ext \quad , \quad S_\lambda := res \circ J_\lambda \circ ext \ . \tag{24}$$

Lemma 5.7 *For* $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$ *the scattering matrix forms a meromorphic family of operators*

$$C^{\pm\infty}(B, V_B(\sigma_\lambda, \varphi)) \to C^{\pm\infty}(B, V_B(\sigma_{-\lambda}, \varphi)) \ .$$

If $\hat{S}_\lambda$ *is singular and* $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$, *then the residue of* $\hat{S}_\lambda$ *is a differential operator.*

Proof. The assertion for the scattering matrix acting on distributions follows from the holomorphy of *ext*, Lemma 5.2, Lemma 5.3, Lemma 4.4, and Fact 3.5. By Corollary 5.4 it restricts nicely to the space of smooth sections. The last assertion is a consequence of Lemma 5.3. □

Lemma 5.8 *If* $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$, *then the adjoint*

$${}^tS_\lambda : C^\infty(B, V_B(\tilde{\sigma}_\lambda, \tilde{\varphi})) \to C^\infty(B, V_B(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))$$

of

$$S_\lambda : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to C^{-\infty}(B, V_B(\sigma_{-\lambda}, \varphi))$$

coincides with the restriction of

$$S_\lambda : C^{-\infty}(B, V_B(\tilde{\sigma}_\lambda, \tilde{\varphi})) \to C^{-\infty}(B, V_B(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))$$

to $C^\infty(B, V_B(\tilde{\sigma}_\lambda, \tilde{\varphi}))$.

Proof. We employ the fact (Lemma 5.5, 5.) that the corresponding relation holds for the intertwining operators (step (25) below). Recall the definition of π^* from the proof of Lemma 4.4. In the domain of convergence of π_* we have

$$\langle \phi, \pi_*(h) \rangle = \sum_{g \in \Gamma} \langle \pi^*(\phi), \pi(g)h \rangle \ ,$$

where $\phi \in C^\infty(B, V_B(\sigma_\lambda, \varphi))$, $h \in C^\infty(\partial X, V(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))$, and $\pi(g) = \pi^{\tilde{\sigma}, -\lambda}(g) \otimes \tilde{\varphi}(g)$. We will use this formula in step (26) below.

Let $\phi \in C^\infty(B, V_B(\sigma_\lambda, \varphi))$, $f \in C^\infty(B, V_B(\tilde{\sigma}_\lambda, \tilde{\varphi}))$, and consider ϕ as a distribution section. Then

$$
\begin{aligned}
\langle \phi, {}^t S_\lambda f \rangle &= \langle S_\lambda \phi, f \rangle \\
&= \langle (res \circ J_\lambda \circ ext)\phi, f \rangle \\
&= \langle (\tilde{res} \circ J_\lambda \circ ext)\phi, f \rangle \\
&= \langle (J_\lambda \circ ext)\phi, \pi^* f \rangle \\
&= \langle ext\phi, ({}^t J_\lambda \circ \pi^*)f \rangle \\
&= \langle ext\phi, (J_\lambda \circ \pi^*)f \rangle & (25) \\
&= \langle \phi, (\pi_* \circ J_\lambda \circ \pi^*)f \rangle \\
&= \sum_{g \in \Gamma} \langle \pi^* \phi, (\pi(g) \circ J_\lambda \circ \pi^*)f \rangle & (26) \\
&= \sum_{g \in \Gamma} \langle \pi^* \phi, (J_\lambda \circ \pi(g) \circ \pi^*)f \rangle \\
&= \sum_{g \in \Gamma} \langle (\pi(g) \circ J_\lambda \circ \pi^*)\phi, \pi^* f \rangle & (27) \\
&= \langle \phi, S_\lambda f \rangle \ . & (28)
\end{aligned}
$$

In order to obtain (28) from (27) we do the transformations backwards with the roles of ϕ and f interchanged. $\qquad \square$

Lemma 5.9 *If $|\mathrm{Re}(\lambda)| < -(\delta_\Gamma + \delta_\varphi)$, then the scattering matrix satisfies the functional equation (viewed as an identity of meromorphic families of operators)*

$$
S_{-\lambda} \circ S_\lambda = \mathrm{id} \ .
$$

Proof. We employ Lemma 4.6, Theorem 4.7, and (23) in order to compute

$$
\begin{aligned}
S_{-\lambda} \circ S_\lambda &= res \circ J_{-\lambda} \circ ext \circ res \circ J_\lambda \circ ext \\
&= res \circ J_{-\lambda} \circ J_\lambda \circ ext \\
&= res \circ ext \\
&= \mathrm{id} \ .
\end{aligned}
$$

$\qquad \square$

The main result of this section is

Theorem 5.10 *Let $X \neq \mathbb{O}H^2$. Then the scattering matrix*

$$
S_\lambda : C^{\pm\infty}(B, V_B(\sigma_\lambda, \varphi)) \to C^{\pm\infty}(B, V_B(\sigma_{-\lambda}, \varphi))
$$

and the extension map

$$ext : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to {}^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \ ,$$

initially defined for $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$, *have meromorphic continuations to all of* $\mathfrak{a}_{\mathbb{C}}^*$. *In particular, we have*

$$ext \quad = \quad J_{-\lambda} \circ ext \circ S_\lambda \ , \tag{29}$$

$$S_{-\lambda} \circ S_\lambda \quad = \quad \mathrm{id} \ . \tag{30}$$

Moreover, ext has at most finite-dimensional singularities.

The proof of the theorem will occupy the remainder of this section. We first use the meromorphic Fredholm theory in order to understand a special case. We introduce the element $\beta \in \mathfrak{a}^*$ as follows

X	$\mathbb{R}H^n$	$\mathbb{C}H^n$	$\mathbb{H}H^n$	$\mathbb{O}H^n$
β	0	0	2α	6α

.

Lemma 5.11 *If* $\delta_\Gamma + \delta_\varphi < 0$ *and* $\hat{M} \ni \sigma = 1$ *is the trivial representation, then* S_λ *and ext have meromorphic continuations to* $W := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \,|\, \mathrm{Re}(\lambda) > -\rho + \beta\}$. *On W the remaining assertions of Theorem 5.10 hold true.*

Proof. We construct the meromorphic continuation of

$$S_\lambda : C^{\infty}(B, V_B(1_\lambda, \varphi)) \to C^{\infty}(B, V_B(1_{-\lambda}, \varphi)) \ ,$$

and then we extend this continuation to distributions by duality using Lemma 5.8. The idea is to set $S_\lambda := S_{-\lambda}^{-1}$ for $\mathrm{Re}(\lambda) < -(\delta_\Gamma + \delta_\varphi)$ and to show that $S_{-\lambda}^{-1}$ forms a meromorphic family on

$$U := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \,|\, \max\{\delta_\Gamma + \delta_\varphi, -\rho + \beta\} < \mathrm{Re}(\lambda) < \rho - \beta\} \ .$$

Let $\{U_\alpha\}$ be a finite open covering of B and let $\tilde{U}_\alpha$ be diffeomorphic lifts of U_α. Choose a subordinated partition of unity $\{\phi_\alpha\}$. Let $\tilde{\phi}_\alpha$ be the corresponding compactly supported function on $\tilde{U}_\alpha$. Then we define $\chi \in {}^{\Gamma}C^{\infty}(\Omega \times \Omega)$ by

$$\chi(x, y) := \sum_\alpha \sum_{g \in \Gamma} \tilde{\phi}_\alpha(gx)\tilde{\phi}_\alpha(gy) \ .$$

Let

$$J_\lambda^{diag} : C^{\infty}(B, V_B(1_\lambda, \varphi)) \to C^{\infty}(B, V_B(1_{-\lambda}, \varphi))$$

be the meromorphic family of operators obtained by multiplying the distribution kernel of $\hat{J}_\lambda$ by χ. If $f \in C^{\infty}(B, V_B(1_\lambda, \varphi))$, then

$$(J_\lambda^{diag})f = \sum_\alpha \tilde{\phi}_\alpha J_\lambda(\tilde{\phi}_\alpha f)$$

using the canonical identifications (see the proof of Lemma 4.4).

For $\lambda \in U$ define

$$R(\lambda) := J_{-\lambda}^{diag} \circ S_\lambda - \mathrm{id} \ . \tag{31}$$

The inverse of the normalized scattering matrix for $\lambda \in U$ should be given by

$$S_\lambda^{-1} = (\mathrm{id} + R(\lambda))^{-1} \circ J_{-\lambda}^{diag} \ . \tag{32}$$

It exists as a meromorphic family if $(\mathrm{id} + R(\lambda))^{-1}$ does.

We apply the meromorphic Fredholm theory (Lemma 3.6) in order to invert $\mathrm{id} + R(\lambda)$ for $\lambda \in U$ and to conclude that $(\mathrm{id} + R(\lambda))^{-1}$ is meromorphic.

We check the assumption of Lemma 3.6. We choose a Hermitian metric on $V_B(1_0, \varphi)$ and a volume form on B. The Hilbert space $\mathcal{H}$ of Proposition 3.6 is $L^2(B, V_B(1_0, \varphi))$ defined using these choices. The Fréchet space $\mathcal{F}$ is just $C^\infty(B, V_B(1_0, \varphi))$. Implicitly, we identify the spaces $C^\infty(B, V_B(1_\lambda, \varphi))$ with $C^\infty(B, V_B(1_0, \varphi))$ using a trivialization of the holomorphic family of bundles $\{V_B(1_\lambda, \varphi)\}_{\lambda \in \mathfrak{a}_\mathbb{C}^*}$.

We need the following well-known fact.

Lemma 5.12 *If* $|\mathrm{Re}(\lambda)| < \rho - \beta$, *then the spherical intertwining operator*

$$J_{1,\lambda} : C^\infty(\partial X, V(1_\lambda)) \to C^\infty(\partial X, V(1_{-\lambda}))$$

is regular and invertible. Moreover, $J_{1,0} = \mathrm{id}$.

Proof. If $|\mathrm{Re}(\lambda)| < \rho - \beta$, then the principal series representation $C^\infty(\partial X, V(1_\lambda))$ is irreducible (see e.g. [29], Ch. VI, Thm. 3.6). By definition of the normalized intertwining operator its restriction to the minimal K-type is regular and bijective. Now the kernel of the leading term in the Laurent expansion of $J_{1,\mu}$ at $\mu = \lambda$ is a subrepresentation of $C^\infty(\partial X, V(1_\lambda))$. It follows that all singular terms in the Laurent expansion vanish and that $J_{1,\lambda}$ is bijective. By Schur's Lemma there is $\mu \in \mathbb{C}$ such that $J_{1,0} = \mu\,\mathrm{id}$. Our normalization of $J_{1,\cdot}$ implies that $\mu = 1$. □

It follows that $J_{-\lambda}^{diag}$ as well as $S_\lambda = res \circ J_\lambda \circ ext$ are regular on U. The difference $J_{-\lambda}^{off} := res \circ J_{-\lambda} - J_{-\lambda}^{diag} \circ res$ is a continuous map from ${}^\Gamma C^{-\infty}(\partial X, V(1_{-\lambda}, \varphi))$ to $C^\infty(B, V_B(1_\lambda, \varphi))$ by Lemma 5.3. Since $R(\lambda) = -J_{-\lambda}^{off} \circ J_\lambda \circ ext$, the family $R(\lambda)$ is a holomorphic family of smoothing operators on U. In addition, $R(0) = 0$. Thus $\mathrm{id} + R(\lambda)$ is invertible at $\lambda = 0$.

We now have verified the assumptions of Lemma 3.6. We conclude that the family S_λ^{-1} is meromorphic family on U with finite-dimensional singularities.

Since $\delta_\Gamma + \delta_\varphi < 0$, we have $-U \cup \{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid \mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi\} = W$. Furthermore, by Lemma 5.9 we have $S_\lambda = S_{-\lambda}^{-1}$ on $-U \cap \{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid \mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi\}$. Thus, setting $S_\lambda := S_{-\lambda}^{-1}$ for $-\lambda \in U$

we obtain a well-defined continuation of S_λ to all of W. By duality this continuation extends to distributions still having finite-dimensional singularities. Moreover, the functional equation (30) holds by definition.

It remains to consider the extension map. We employ the scattering matrix in order to define for $\lambda \in W$, $\text{Re}(\lambda) < -(\delta_\Gamma + \delta_\varphi)$

$$ext_1 := J_{-\lambda} \circ ext \circ S_\lambda \ .$$

We claim that $ext = ext_1$. In fact, since res is injective on $\{|\text{Re}(\lambda)| < -(\delta_\Gamma + \delta_\varphi)\}$ by Theorem 4.7, the computation

$$res \circ ext_1 = res \circ J_{-\lambda} \circ ext \circ S_\lambda = S_{-\lambda} \circ S_\lambda = \text{id}$$

and Lemma 4.5 imply the claim.

We now have constructed a meromorphic continuation of ext to all of W. The relation (29) between the scattering matrix and ext follows by meromorphic continuation. This equation also implies that ext has at most finite-dimensional singularities. Thus the proof of Lemma 5.11 is complete. $\square$

In the next step we drop the assumption $\delta_\Gamma + \delta_\varphi < 0$ employing the fact that for $X \neq \mathbb{O}H^2$ the symmetric space X belongs to a series.

Lemma 5.13 *If $X \neq \mathbb{O}H^2$, then Lemma 5.11 holds true without the assumption $\delta_\Gamma + \delta_\varphi < 0$.*

Proof. X belongs to a series of rank-one symmetric spaces. Let $\ldots \subset G^n \subset G^{n+1} \subset \ldots$ be the corresponding sequence of real, semisimple, linear Lie groups inducing embeddings of the corresponding Iwasawa constituents $K^n \subset K^{n+1}$, $N^n \subset N^{n+1}$, $M^n \subset M^{n+1}$ such that $A = A^n = A^{n+1}$. Then we have totally geodesic embeddings of the symmetric spaces $X^n \subset X^{n+1}$ inducing embeddings of their boundaries $\partial X^n \subset \partial X^{n+1}$. If $\Gamma \subset G^n$ is convex-cocompact then it is still convex-cocompact viewed as a subgroup of G^{n+1}. We obtain embeddings $\Omega^n \subset \Omega^{n+1}$ inducing $B^n \subset B^{n+1}$ while the limit set Λ^n is identified with Λ^{n+1}. Let $\rho^n(H) = \frac{1}{2}\text{tr}(\text{ad}(H)_{|\mathfrak{n}^n})$, $H \in \mathfrak{a}$.

The exponent of Γ now depends on n and is denoted by δ_Γ^n. We have the relation $\delta_\Gamma^{n+1} = \delta_\Gamma^n - \zeta$, where $\zeta := \rho^{n+1} - \rho^n > 0$. Thus $\delta_\Gamma^{n+m} \to -\infty$ as $m \to \infty$. Hence, taking m large enough we obtain $\delta_\Gamma^{n+m} + \delta_\varphi < 0$. The aim of the following discussion is to show how the meromorphic continuation of ext^{n+1} leads to the continuation of ext^n.

Let $P^n := M^n A^n N^n$, $V(1_\lambda, \varphi)^n := G^n \times_{P^n} V_{1_\lambda} \otimes V_\varphi$, and $V_{B^n}(1_\lambda, \varphi) = \Gamma\backslash V(1_\lambda, \varphi)^n_{|\Omega^n}$. Here as always (φ, V_φ) is a finite-dimensional representation representation of Γ. The representation V_{1_λ} of P^{n+1} restricts to the representation $V_{1_{\lambda-\zeta}}$ of P^n. This induces isomorphisms of bundles

$$V(1_\lambda, \varphi)^{n+1}_{|\partial X^n} \cong V(1_{\lambda-\zeta}, \varphi)^n, \quad V_{B^{n+1}}(1_\lambda, \varphi)_{|B^n} \cong V_{B^n}(1_{\lambda-\zeta}, \varphi) \ .$$

Let

$$i^* : C^\infty(B^{n+1}, V_{B^{n+1}}(1_\lambda, \tilde\varphi)) \quad \to \quad C^\infty(B^n, V_{B^n}(1_{\lambda-\zeta}, \tilde\varphi)) \,,$$
$$j^* : C^\infty(\partial X^{n+1}, V(1_\lambda, \tilde\varphi)^{n+1}) \quad \to \quad C^\infty(\partial X^n, V(1_{\lambda-\zeta}, \tilde\varphi)^n)$$

denote the maps given by restriction of sections. Note that j^* is G^n-equivariant. The adjoint maps define the push-forward of distribution sections

$$i_* : C^{-\infty}(B^n, V_{B^n}(1_\lambda, \varphi)) \quad \to \quad C^{-\infty}(B^{n+1}, V_{B^{n+1}}(1_{\lambda-\zeta}, \varphi)) \,,$$
$$j_* : C^{-\infty}(\partial X^n, V(1_\lambda, \varphi)^n) \quad \to \quad C^{-\infty}(\partial X^{n+1}, V(1_{\lambda-\zeta}, \varphi)^{n+1}) \,.$$

If $\phi \in C^{-\infty}(B^n, V_{B^n}(1_\lambda, \varphi))$, then the push forward $i_*\phi$ has support in $B^n \subset B^{n+1}$. Since $res^{n+1} \circ ext^{n+1} = \text{id}$ we have $\text{supp}(ext^{n+1} \circ i_*)(\phi) \subset \Lambda^{n+1} \cup \Omega^n = \partial X^n$.

Assume that ext^{n+1} is meromorphic on $W^{n+1} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \text{Re}(\lambda) > -\rho^{n+1} + \beta\}$. We are now going to continue ext^n using i_*, ext^{n+1} and a left inverse of j_*. As in previous occasions we trivialize the family $\{V(1_\lambda)^{n+1}\}_\lambda$. We identify $C^\infty(\partial X^{n+1}, V(1_\lambda, \tilde\varphi)^{n+1})$ with $C^\infty(\partial X^{n+1}) \otimes \tilde{V}_\varphi$ for all $\lambda \in \mathbb{C}$. Let $U \subset \partial X^{n+1}$ be a tubular neighbourhood of ∂X and fix a diffeomorphism $T : (-1, 1) \times \partial X^n \xrightarrow{\cong} U$. Let $\chi \in C_c^\infty((-1, 1))$ be a cut-off function satisfying $\chi(0) = 1$. Then we define a continuous extension $t : C^\infty(\partial X^n) \otimes V_{\tilde\varphi} \to C^\infty(\partial X^{n+1}) \otimes V_{\tilde\varphi}$ by

$$tf(x) := \begin{cases} \chi(r)f(x') & \text{if } x = T(r, x') \in U, \quad r \in (-1, 1), \ x' \in \partial X^n \\ 0 & \text{if } x \notin U \end{cases} \,.$$

Let $t' : C^{-\infty}(\partial X^{n+1}, V(1_{\lambda-\zeta}, \varphi)^{n+1}) \to C^{-\infty}(\partial X^n, V(1_\lambda, \varphi)^n)$ be the adjoint of t. Then $t' \circ j_* = \text{id}$. Now we can define

$$\widetilde{ext}^n \phi := (t' \circ ext^{n+1} \circ i_*)(\phi) \,.$$

Then

$$\widetilde{ext}^n : C^{-\infty}(B^n, V_{B^n}(1_\lambda, \varphi)) \to C^{-\infty}(\partial X^n, V(1_\lambda, \varphi)^n)$$

is a meromorphic family on $W^{n+1} + \zeta = W^n$ of continuous maps with at most finite-dimensional singularities.

In order to prove that $\widetilde{ext}^n$ provides the desired meromorphic continuation it remains to show that it coincides with ext^n in the region $\text{Re}(\lambda) > \delta_\Gamma^n + \delta_\varphi$. If $\text{Re}(\lambda) > \delta_\Gamma^n + \delta_\varphi$, then $\text{Re}(\lambda) - \zeta > \delta_\Gamma^{n+1} + \delta_\varphi$, and the push-down maps $\pi_{*,-\lambda}^n$, $\pi_{*,-\lambda+\zeta}^{n+1}$ are defined. It is easy to see from the definition of the push-down that in the domain of convergence

$$i^* \circ \pi_*^{n+1} = \pi_*^n \circ j^* \,.$$

Taking adjoints we obtain $ext^{n+1} \circ i_* = j_* \circ ext^n$. Therefore we have

$$\widetilde{ext}^n = t' \circ ext^{n+1} \circ i_* = t' \circ j_* \circ ext^n = ext^n \,.$$

It follows by meromorphy that $\text{im}(\widetilde{ext}^n)$ consists of Γ-invariant sections for all $\lambda \in W^n$.

We define the meromorphic continuation of the scattering matrix by (24). Then it is easy to see that the scattering matrix has the properties as asserted. This finishes the proof of Lemma

5.13. $\square$

We now use tensoring with finite-dimensional G-representations in order to complete the proof of Theorem 5.10. For a moment let $\sigma \in \hat{M}$. Then the theory of highest weights for G implies that there are sequences $\mu_i \in \mathfrak{a}^*$, $\mu_i \to \infty$ and π_{σ,μ_i} of finite-dimensional irreducible representations of G such that π_{σ,μ_i} has highest $\mathfrak{a}$-weight μ_i, and the representation of M on the corresponding highest weight space is equivalent to σ. More details on this can be found e.g. in [11], pp. 39-41. If $\sigma = \sigma' \oplus \sigma'^w$ for some non-Weyl-invariant $\sigma' \in \hat{M}$, then we set $\pi_{\sigma,\mu_i} := \pi_{\sigma',\mu_i} \oplus \pi_{\sigma'^w,\mu_i}$. As in the proof of Theorem 4.7, Equation (14), we obtain an embedding of bundles

$$V(\sigma_\lambda, \varphi) \hookrightarrow V(1_{\lambda+\mu_i}, \varphi \otimes \pi_{\sigma,\mu_i}) \tag{33}$$

as well as an injective, local, Γ-intertwining operator

$$i_{\sigma,\mu_i} : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \hookrightarrow C^{-\infty}(\partial X, V(1_{\lambda+\mu_i}, \varphi \otimes \pi_{\sigma,\mu_i})) \ . \tag{34}$$

It induces the embedding

$$i^B_{\sigma,\mu_i} : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \hookrightarrow C^{-\infty}(B, V_B(1_{\lambda+\mu_i}, \varphi \otimes \pi_{\sigma,\mu_i})) \ .$$

Let i_{σ,μ_i}' and $i^B_{\sigma,\mu_i}{}'$ be the adjoint operators, i.e., the projections onto the spaces smooth sections of the corresponding dual bundles. Then in the domain of convergence of the push-down map $\mathrm{Re}(\lambda) < -(\delta_\Gamma + \delta_\varphi)$ we have

$$i^B_{\sigma,\mu_i}{}' \circ \pi_* = \pi_* \circ i_{\sigma,\mu_i}' \ .$$

Note that the domains of convergence of both sides coincide. It follows that for $\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi$, $\phi \in C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$,

$$ext \circ i^B_{\sigma,\mu_i}(\phi) = i_{\sigma,\mu_i} \circ ext(\phi) \ . \tag{35}$$

Now let $\nu \in \mathfrak{a}*$ be arbitrary, and let $W_\nu := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \mathrm{Re}(\lambda) > \nu\}$ be the corresponding half-plane. Choose μ_i large enough such that $\mathrm{Re}(\lambda) + \mu_i > -\rho + \beta$ for all $\lambda \in W_\nu$. By Lemma 5.13 the extension map on the left hand side of (35) has a meromorphic continuation to W_ν with finite-dimensional singularities. Moreover,

$$ext \circ i^B_{\sigma,\mu_i} \left(C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \right) \subset i_{\sigma,\mu_i} \left(C^{-\infty}(\partial X, V(1_{\lambda+\mu_i}, \varphi \otimes \pi_{\sigma,\mu_i})) \right) \ .$$

In fact, this is true for λ in the domain of convergence by (35), hence on all of W_ν by meromorphy. Therefore we can define for $\phi \in C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$, $\lambda \in W_\nu$

$$ext(\phi) := (i_{\sigma,\mu_i})^{-1} \circ ext \circ i^B_{\sigma,\mu_i}(\phi) \ .$$

This gives the desired meromorphic continuation of ext to W_ν. Now (35) holds on all of W_ν. It follows that the singularities of ext are at most finite-dimensional. Since ν was arbitrary ext is meromorphic on all of $\mathfrak{a}^*_{\mathbb{C}}$.

We define the meromorphic continuation of the scattering matrix by (24). Using Lemma 4.6 and Theorem 4.7 it is easy to verify the functional equations (30) and (29) for $\mathrm{Re}(\lambda) < -(\delta_\Gamma + \delta_\varphi)$. By meromorphy they hold on all of $\mathfrak{a}^*_{\mathbb{C}}$. This finishes the proof of Theorem 5.10. $\square$

6 Invariant distributions on the limit set

In the present section we study the spaces $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ of invariant distributions which are supported on the limit set. The main result of this section is

Theorem 6.1 *Let $X \neq \mathbb{O}H^2$. Fix finite-dimensional representations σ of M and φ of Γ. Then*

1. *The set $\{\lambda \in \mathfrak{a}_\mathbb{C}^* \mid {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \neq 0\}$ is discrete.*

2. *For each $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the space $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ is finite-dimensional.*

The proof is based on the following observation: Assume that ext is singular at $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Since $res \circ ext(\phi_\mu) = \phi_\mu$ is regular for any holomorphic family $\mu \mapsto \phi_\mu \in C^{-\infty}(B, V_B(\sigma_\mu, \varphi))$ the leading singular part of the Laurent expansion of $ext(\phi_\mu)$ at $\mu = \lambda$ belongs to $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. We show that for almost all λ these spaces are generated by the singular parts of ext.

First we need detailed knowledge of the asymptotics of the Poisson tansform.

Lemma 6.2 *Let γ be a finite-dimensional representation of K, $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$, and let $w \in N_K(\mathfrak{a})$ represent the non-trivial element of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$.*

1. *Let $f \in C^\infty(\partial X, V(\sigma_\lambda))$. If $\mathrm{Re}(\lambda) > 0$, then there exists $\epsilon > 0$ (depending on λ but not on f) such that for $a \to \infty$ we have*

$$(P_\lambda^T f)(ka) = a^{\lambda - \rho} c_\gamma(\lambda) T f(k) + O(a^{\mathrm{Re}(\lambda) - \rho - \epsilon})$$

uniformly in $k \in K$. If $|\mathrm{Re}(\lambda)| < \frac{1}{2}|\alpha|$ and $\lambda \neq 0$, then

$$(P_\lambda^T f)(ka) = a^{\lambda - \rho} c_\gamma(\lambda) T f(k) + a^{-\lambda - \rho} T^w(\hat{J}_\lambda f)(k) + O(a^{-\frac{\alpha}{2} - \rho - \epsilon}) \tag{36}$$

uniformly in $k \in K$. The remainder depends jointly continuously on λ and f.

If $\hat{J}_0$ is regular, then (36) remains valid for $\lambda = 0$. Otherwise we have

$$\begin{aligned} (P_0^T f)(ka) &= \log(a) a^{-\rho} \left(2\mathrm{res}_{\lambda=0}(c_\gamma(\lambda)) T f(k)\right) \\ &\quad + a^{-\rho} \left(c_\gamma^0 T f(k) + T^w(\hat{J}_0^0 f)(k)\right) + O(a^{-\rho - \alpha}), \end{aligned}$$

where c_γ^0, $\hat{J}_0^0$ denote the constant terms in the Laurent expansions of c_γ and $\hat{J}_\lambda$ at $\lambda = 0$.

2. *Let $\partial X = U \cup Q$, where U is open and $Q := \partial X \setminus U$. Let $f \in C^{-\infty}(\partial X, V(\sigma_\lambda))$ with $\mathrm{supp} f \subset Q$. Then there exist smooth functions ψ_n, $n \in \mathbb{N}$, on U such that*

$$(P_\lambda^T f)(ka) = a^{-(\lambda + \rho)} T^w(\hat{J}_\lambda f)(k) + \sum_{n \geq 1} a^{-(\lambda + \rho) - n\alpha} \psi_n(k), \quad k \in U. \tag{37}$$

The series converges uniformly for $a \gg 0$ and k in compact subsets of U. In particular, for $a \to \infty$ we have

$$(P_\lambda^T f)(ka) = a^{-\lambda-\rho} T^w (\hat{J}_\lambda f)(k) + O(a^{-\lambda-\rho-\alpha})$$

uniformly as kM varies in compact subsets of U.

3. *Let U, Q be as in 2. and $f \in C^{-\infty}(\partial X, V(\sigma_\lambda))$ such that $\mathrm{res}_U f \in C^\infty(U, V(\sigma_\lambda))$. If $\mathrm{Re}(\lambda) > 0$, then there exists $\epsilon > 0$ such that for $a \to \infty$ we have*

$$(P_\lambda^T f)(ka) = a^{\lambda-\rho} c_\gamma(\lambda) T f(k) + O(a^{\lambda-\rho-\epsilon})$$

uniformly as kM varies in compact subsets of U. If $|\mathrm{Re}(\lambda)| < \frac{1}{2}|\alpha|$, $\lambda \neq 0$, then we have for $a \to \infty$

$$(P_\lambda^T f)(ka) = a^{\lambda-\rho} c_\gamma(\lambda) T f(k) + a^{-\lambda-\rho} T^w (\hat{J}_\lambda f)(k) + O(a^{-\frac{\alpha}{2}-\rho-\epsilon})$$

uniformly as kM varies in compact subsets of U. The remainder depends jointly continuously on λ and f.

The asymptotic expansions can be differentiated with respect to a.

Proof. Assertion 1 is a consequence of the general results concerning the asymptotics of matrix coefficients of admissible representations including their dependence on parameters ([54], Thm. 4.4.3, [55], 12.4., 12.5., 12.6., [42]) combined with the limit formulas for the Poisson transform (see [53] or [42], also [54], Thm. 5.3.4)

$$\lim_{a\to\infty} a^{\rho-\lambda}(P_\lambda^T f)(ka) = c_\gamma(\lambda) T f(k), \qquad \mathrm{Re}(\lambda) > 0 , \tag{38}$$

$$\lim_{a\to\infty} a^{\rho+\lambda}(P_\lambda^T f)(ka) = T^w (\hat{J}_\lambda f)(k), \qquad \mathrm{Re}(\lambda) < 0 . \tag{39}$$

3. is a consequence of 1. and 2.. Indeed let $W, W_1 \subset U$ be a compact subsets such that $W \subset \mathrm{int}(W_1)$. Let $\chi \in C_c^\infty(U)$ be such that $\chi_{|W_1} \equiv 1$. Then we can write $f = \chi f + (1-\chi)f$, where χf is smooth and $\mathrm{supp}(1-\chi)f \subset \partial X \setminus \mathrm{int}(W_1)$. We now apply 1. to χf and 2. to $(1-\chi)f$ for $kM \in W$.

It remains to prove assertion 2. We imitate the argument of [6], Thm. 4.8. which proves the assertion for the case $\sigma = 1$.

Let $w \in N_K(\mathfrak{a})$ be a representative of the non-trivial element of $W(\mathfrak{g}, \mathfrak{a})$. In the following computation we write the pairing of a distribution with a smooth function as an integral.

$$\begin{aligned}
(P_\lambda^T f)(ka) &= \int_K a(a^{-1}k^{-1}h)^{-(\lambda+\rho)} \gamma(\kappa(a^{-1}k^{-1}h)) T f(h) dh \\
&= \int_{\bar{N}} a(a^{-1}w\kappa(\bar{n}))^{-(\lambda+\rho)} \gamma(\kappa(a^{-1}w\kappa(\bar{n}))) T f(kw\kappa(\bar{n})) a(\bar{n})^{-2\rho} d\bar{n}
\end{aligned}$$

$$= \int_{\bar{N}} a(a\bar{n}a^{-1})^{-(\lambda+\rho)} a^{-(\lambda+\rho)} a(\bar{n})^{\lambda+\rho} \gamma(w)\gamma(\kappa(a\bar{n}a^{-1})) Tf(kw\kappa(\bar{n}))a(\bar{n})^{-2\rho} d\bar{n}$$

$$= a^{-(\lambda+\rho)}\gamma(w) \int_{\bar{N}} a(\bar{n})^{\lambda-\rho} a(a\bar{n}a^{-1})^{-(\lambda+\rho)} \gamma(\kappa(a\bar{n}a^{-1})) Tf(kw\kappa(\bar{n})) d\bar{n} . \tag{40}$$

For $z \in \mathbb{R}^+$ define $a_z \in A$ by $z = a_z^{-\alpha}$. We consider the map $\Phi : (0,\infty) \times \bar{N} \ni (z,\bar{n}) \mapsto a_z \bar{n} a_z^{-1} \in \bar{N}$ which according to the decomposition of $\mathfrak{n}$ into root spaces $\mathfrak{n} = \mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha}$ can also be written as

$$\Phi(z, \exp(X+Y)) := \exp(zX + z^2 Y), \quad X \in \mathfrak{n}_\alpha, Y \in \mathfrak{n}_{2\alpha} .$$

Thus Φ and hence $(z,\bar{n}) \mapsto a(a_z\bar{n}a_z^{-1})^{-(\lambda+\rho)}\gamma(\kappa(a_z\bar{n}a_z^{-1}))$ extend analytically to $\mathbb{R} \times \bar{N}$. Taking the Taylor expansion with respect to z at $z = 0$ we obtain

$$a(a_z\bar{n}a_z^{-1})^{-(\lambda+\rho)}\gamma(\kappa(a_z\bar{n}a_z^{-1})) = \mathrm{id} + \sum_{n \geq 1} A_n(\bar{n})z^n .$$

Here $A_n : \bar{N} \to \mathrm{End}(V_\gamma)$ are analytic and the series converges in the spaces of smooth functions on $\bar{N}$ with values in $\mathrm{End}(V_\gamma)$.

Inserting this expansion into (40) we obtain

$$(P_\lambda^T f)(ka) = a^{-(\lambda+\rho)} T^w(\hat{J}_\lambda f)(k) + \sum_{n \geq 1} a^{-(\lambda+\rho)-n\alpha} \psi_n(k) ,$$

where

$$\psi_n(k) := \gamma(w) \int_{\bar{N}} A_n(\bar{n}) T a(\bar{n})^{\lambda-\rho} f(kw\kappa(\bar{n})) d\bar{n} .$$

Note that $k \mapsto f(kw\kappa(.))$ is a smooth family of distributions with compact support in $\bar{N}$. Thus ψ_n is smooth. This finishes the proof of the lemma. $\qquad\square$

Corollary 6.3 *Let $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda))$ and $f \in C^\infty(\partial X, V((\tilde{\gamma}_{|M})_{-\lambda}))$. If $\mathrm{Re}(\lambda) > 0$, then there exists $\epsilon > 0$ such that for $a \to \infty$ we have*

$$\int_K \langle P_\lambda^T \phi(ka), f(k)\rangle \, dk = a^{\lambda-\rho} \langle \phi, {}^t(c_\gamma(\lambda)T)f\rangle + O(a^{\lambda-\rho-\epsilon}) .$$

There are corresponding formulas for $\mathrm{Re}(\lambda) = 0$.

Proof. The argument is adapted from [51], 5.1. Let v_i be a basis of $V_{\tilde{\gamma}}$, and write $f(k) = \sum f_i(k)v_i$. Define $\check{} : C^\infty(K) \to C^\infty(K)$ by $\check{h}(k) := h(k^{-1})$. Let $*$ be the convolution on the group K. Then we have

$$\begin{aligned}
\int_K \langle P_\lambda^T \phi(ka), f(k)\rangle \, dk &= \sum_i \langle v_i, \int_K P_\lambda^T \phi(ka) f_i(k) \, dk\rangle \\
&= \sum_i \langle v_i, (\check{f}_i * P_\lambda^T \phi(.a))(1)\rangle \\
&= \sum_i \langle v_i, P_\lambda^T(\check{f}_i * \phi)(a)\rangle .
\end{aligned}$$

Now observe that $\check{f}_i * \phi \in C^\infty(\partial X, V(\sigma_\lambda))$ and apply Lemma 6.2.   □

Now we show a variant of Green's formula. We need nice cut-off functions which exist by the following lemma. Let Δ_X be the Laplace-Beltrami operator of X.

Lemma 6.4 *There exists a cut-off function* $\chi \in C_c^\infty(X \cup \Omega)$ *such that*

1. $\sum_{g \in \Gamma} g^*\chi = 1$,

2. $\sup_{gK \in X} a_g^\alpha \, |d\chi(gK)| < \infty$,

3. $\sup_{gK \in X} a_g^\alpha \, |\Delta_X \chi(gK)| < \infty$,

4. $\sup_{gK \in X} |\chi(DgK)| < \infty$, $\forall D \in \mathcal{U}(\mathfrak{g})$.

We denote the restriction of χ *to* Ω *by* χ_∞.

Proof.   Let $W \subset X \cup \Omega$ be a compact subset such that $\bigcup_{g \in \Gamma} gW = X \cup \Omega$. Then we choose a cut-off function $\psi \in C_c^\infty(X \cup \Omega)$ such that $\psi_{|W} = 1$. We define

$$\chi := \frac{\psi}{\sum_{g \in \Gamma} g^*\psi} \ .$$

Note that χ is well-defined since $\sum_{g \in \Gamma} g^*\psi$ never vanishes on $X \cup \Omega$. Property 1 is obvious by the definition of χ.

In the following we consider χ as an element of $C^\infty(\bar{X})$. Since G acts smoothly on the compact manifold $\bar{X}$ we have $L^\sharp \in C^\infty(\bar{X}, T\bar{X})$, where $L^\sharp$ denotes the fundamental vector field corresponding to $L \in \mathfrak{g}$. If $Y \in C^\infty(\bar{X}, T\bar{X})$, then $Y(C^\infty(\bar{X})) \subset C^\infty(\bar{X})$. Thus the left action of $\mathfrak{g}$ and hence of $\mathcal{U}(\mathfrak{g})$ preserves $C^\infty(\bar{X})$. This shows assertion 4.

We have $d\chi \in C^\infty(\bar{X}, T^*\bar{X})$. The tensor field in $C^\infty(X, S^2 TX)$ which is dual to the Riemannian metric of X has a continuous extension to $\bar{X}$ vanishing of second order at ∂X. We conclude that $|d\chi|$ vanishes of first order at the boundary of $\bar{X}$. This shows assertion 2.

Since the coefficients of Δ_X vanish at ∂X of at least first order assertion 3 follows.   □

Let $\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. Then $res \circ \hat{J}_\lambda(\phi) \in C^\infty(B, V_B(\sigma_{-\lambda}, \varphi))$ is well-defined even if $\hat{J}_\mu$ has a pole at $\mu = \lambda$. In the latter case the residue of $\hat{J}_\mu$ at $\mu = \lambda$ is a differential operator D_λ (see Lemma 5.3) and $res \circ D_\lambda(\phi) = 0$.

Proposition 6.5 *If* $\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ *and* $f \in {}^\Gamma C^{-\infty}(\partial X, V(\tilde{\sigma}_\lambda, \tilde{\varphi}))$, *then*

$$\langle res \circ \hat{J}_\lambda(\phi), res(f) \rangle = 0 \ .$$

Proof. At first we need

Lemma 6.6 *The space*

$$^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) := \{f \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \mid f_{|\Omega} \in C^\infty(\Omega, V(\sigma_\lambda, \varphi))\}$$

is dense in $^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$.

Proof. By Theorem 5.10 *ext* has an at most finite-dimensional singularity at λ. Thus there is a finite-dimensional subspace $W \subset C^\infty(B, V_B(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))$ such that

$$ext_{|W^\perp} : W^\perp \to {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$$

is a well-defined continuous map, where $W^\perp := \{\phi \in C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \mid \langle \phi, W \rangle = \{0\}\}$. Since $C^\infty(B, V_B(\sigma_\lambda, \varphi)) \subset C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ is dense we can choose a complement $\tilde{W} \subset C^\infty(B, V_B(\sigma_\lambda, \varphi))$ such that $C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) = W^\perp \oplus \tilde{W}$.

Let $f \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. Then we can write $res(f) = g = g^\perp \oplus \tilde{g}$, $g^\perp \in W^\perp$, $\tilde{g} \in \tilde{W}$. Now $res(f - ext(g^\perp)) = \tilde{g}$. It follows that $f - ext(g^\perp) \in {}^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. Let now $g_i \in C^\infty(B, V_B(\sigma_\lambda, \varphi))$ be a sequence such that $\lim_{i\to\infty} g_i = g$. Then we can decompose $g_i = g_i^\perp + \tilde{g}_i$. The sections $g_i^\perp$ are smooth since g_i and $\tilde{g}_i \in \tilde{W}$ are so. It follows that $ext(g_i^\perp) \in {}^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. By continuity of $ext_{|W^\perp}$ we have $ext(g^\perp) = \lim_{i\to\infty} ext(g_i^\perp)$. The assertion of the lemma now follows from $f = f - ext(g^\perp) + \lim_{i\to\infty} ext(g_i^\perp)$. $\square$

We now prove Proposition 6.5 for the case $\sigma = 1$, $\mathrm{Re}(\lambda) > 0$. We consider the Poisson transforms, both denoted by P,

$$P = P_\lambda \otimes \mathrm{id} : C^{-\infty}(\Lambda, V(1_\lambda, \varphi)) \to C^\infty(X, V(1, \varphi))$$

and

$$P = P_\lambda \otimes \mathrm{id} : C^{-\infty}(\Lambda, V(1_\lambda, \tilde{\varphi})) \to C^\infty(X, V(1, \tilde{\varphi})) .$$

Let $D := \Delta_X \otimes \mathrm{id} - \rho^2 + \lambda^2$ be the shifted Laplace operator acting on $C^\infty(X, V(1, \varphi))$ and $C^\infty(X, V(1, \tilde{\varphi}))$, respectively. Then we have $D \circ P = 0$. Let χ be a cut-off function as in Lemma 6.4. By B_R we denote the metric R-ball centered at the origin of X. The following is an application of Green's formula:

$$\begin{aligned}
0 &= \int_{B_R} \chi(x) \left(\langle DP\phi(x), Pf(x) \rangle - \langle P\phi(x), DPf(x) \rangle \right) \, dx \\
&= \int_{B_R} \left(\langle D\chi(x)P\phi(x), Pf(x) \rangle - \langle \chi(x)P\phi(x), DPf(x) \rangle - \langle [D, \chi]P\phi(x), Pf(x) \rangle \right) \, dx \\
&= -\int_{\partial B_R} \left(\langle \nabla_n \chi(y)P\phi(y), Pf(y) \rangle - \langle \chi(y)P\phi(y), \nabla_n Pf(y) \rangle \right) \, dy \\
&\quad - \int_{B_R} \langle [D, \chi]P\phi(x), Pf(x) \rangle \, dx ,
\end{aligned} \tag{41}$$

where n is the exterior unit normal vector field at ∂B_R.

By Lemma 6.6 we can assume that $f_{|\Omega}$ is smooth. Then Lemma 6.2, 2. and 3. combined with properties 2 and 3 of χ implies that $|\langle [D, \chi] P\phi, Pf \rangle|$ is integrable over all of X. From Lemma 6.4, property 1, and the Γ-invariance of $\langle P\phi, Pf \rangle$ it follows that $\int_X \langle [D, \chi] P\phi, Pf \rangle \, dx = 0$. Taking the limit $R \to \infty$ in (41) we obtain by Lemma 6.2, 2. and 3.

$$
\begin{aligned}
0 &= (\lambda + \rho) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_\lambda \phi)(k), c_1(\lambda) f(k) \rangle \, dk \\
&\quad + (\lambda - \rho) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_\lambda \phi)(k), c_1(\lambda) f(k) \rangle \, dk \\
&= 2\lambda c_1(\lambda) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_\lambda \phi)(k), f(k) \rangle \, dk \\
&= 2\lambda c_1(\lambda) \langle res \circ \hat{J}_\lambda(\phi), res(f) \rangle \ .
\end{aligned}
$$

This is the assertion of the proposition in our special case since $c_1(\lambda) \neq 0$ for $\mathrm{Re}(\lambda) > 0$.

Note that almost the same proof would also work for general σ and $\mathrm{Re}(\lambda) \geq 0$. But for $\mathrm{Re}(\lambda) < 0$ we have to employ tensoring with finite-dimensional representations, anyway. This method will reduce matters to the case $\sigma = 1$, $\mathrm{Re}(\lambda) > 0$ treated above. As in the proof of Theorem 5.10 we consider the finite-dimensional representation $\pi_{\sigma,\mu}$ of G and the embedding

$$
i_{\sigma,\mu} : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \hookrightarrow C^{-\infty}(\partial X, V(1_{\lambda+\mu}, \varphi \otimes \pi_{\sigma,\mu})) \ .
$$

In addition, the projection onto the lowest weight space of $\pi_{\sigma,\mu}$ induces a surjection

$$
p_{\sigma,\mu} : C^{-\infty}(\partial X, V(1_{-\lambda-\mu}, \varphi \otimes \pi_{\sigma,\mu})) \hookrightarrow C^{-\infty}(\partial X, V(\sigma_{-\lambda}, \varphi)) \ .
$$

Then we have the following identity of meromorphic families of operators (see [55], 10.2.6)

$$
\hat{J}_{\sigma,\lambda} \otimes \mathrm{id}_{V_\varphi} = p_{\sigma,\mu} \circ \left(\hat{J}_{1,\lambda+\mu} \otimes \mathrm{id}_{V_\varphi \otimes V_{\pi_{\sigma,\mu}}} \right) \circ i_{\sigma,\mu} \ . \tag{42}
$$

We also consider the induced operators on B denoted by $i_{\sigma,\mu}^B$ and $p_{\sigma,\mu}^B$. Now let ϕ and f be as in the statement of the proposition. Choose $\mu \in \mathfrak{a}^*$ large enough such that $\pi_{\sigma,\mu}$ exists and $\mathrm{Re}(\lambda + \mu) > 0$. Then we have by the first part of the proof

$$
\begin{aligned}
\langle res \circ \hat{J}_\lambda(\phi), res(f) \rangle &= \langle res \circ p_{\sigma,\mu} \circ \hat{J}_{\lambda+\mu} \circ i_{\sigma,\mu}(\phi), res(f) \rangle \\
&= \langle p_{\sigma,\mu}^B \circ res \circ \hat{J}_{\lambda+\mu} \circ i_{\sigma,\mu}(\phi), res(f) \rangle \\
&= \langle res \circ \hat{J}_{\lambda+\mu} \circ i_{\sigma,\mu}(\phi), i_{\tilde{\sigma},\mu}^B \circ res(f) \rangle \\
&= \langle res \circ \hat{J}_{\lambda+\mu} \circ i_{\sigma,\mu}(\phi), res \circ i_{\tilde{\sigma},\mu}(f) \rangle \\
&= 0 \ .
\end{aligned}
$$

This proves the proposition. $\square$

Let $\mathfrak{m}$ denote the Lie algebra of M. We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$. Then $\mathfrak{t} \oplus \mathfrak{a} =: \mathfrak{h}$ is a Cartan algebra of $\mathfrak{g}$. Via the Harish-Chandra isomorphism characters of $\mathcal{Z}$ are parametrized

by elements of $\mathfrak{h}_{\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, where $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the Weyl group of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. A character χ_ν, $\nu \in \mathfrak{h}_{\mathbb{C}}^*$, is called integral, if

$$2\frac{\langle \nu, \varepsilon \rangle}{\langle \varepsilon, \varepsilon \rangle} \in \mathbb{Z} \tag{43}$$

for all roots ε of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

We further choose a positive root system of $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_C)$. Let ρ_m denote half of the sum of these positive roots. For $\sigma \in \hat{M}$ let $\mu_\sigma \in i\mathfrak{t}^*$ be its highest weight. The infinitesimal character of the principal series representation $\pi^{\sigma,\lambda}$ of G, $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, is now given by $\chi_{\mu_\sigma + \rho_m - \lambda}$. If $X \neq \mathbb{R}H^2$, then for $\sigma \in \hat{M}$ we define the lattice

$$I_\sigma := \{\lambda \in \mathfrak{a}^* \mid \chi_{\mu_\sigma + \rho_m - \lambda} \text{ is integral} \} .$$

If $X = \mathbb{R}H^2$ and $G = SL(2, \mathbb{R})$, then $M \cong \mathbb{Z}_2$. Let ± 1 denote the trivial $(+)$, resp. non-trivial $(-)$ irreducible representation of M. We define

$$I_1 := (\frac{1}{2} + \mathbb{Z})\alpha, \quad I_{-1} := \mathbb{Z}\alpha .$$

If $G = PSL(2, \mathbb{R})$, then $M = \{1\}$, and we define $I_1 := (\frac{1}{2} + \mathbb{Z})\alpha$.

If $\lambda \notin I_\sigma$, then the principal series representation $\pi^{\sigma,\lambda}$ is irreducible (see [17], 4.3.3). Note that $I_{\sigma^w} = I_\sigma$, so the definition is compatible with our previous convention concerning the Weyl-invariance of σ. Let $I_\mathfrak{a} \subset \mathfrak{a}^*$ be the $\mathbb{Z}$-module generated by the short root α, if 2α is a root, or by $\alpha/2$, if not. Then all poles of $\hat{J}_\lambda$ are located in $I_\mathfrak{a}$ ([32], Thm. 3 and Prop. 43). Note that for any σ we have that $I_\sigma \subset I_\mathfrak{a}$ is a sublattice of index 2.

Lemma 6.7 *Assume that $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfy one of the following conditions*

1. *$\text{Re}(\lambda) \geq 0$ and $\sigma = 1$.*

2. *$\text{Re}(\lambda) \geq 0$, $G = SL(2, \mathbb{R})$ and $\sigma = -1$.*

3. *$\text{Re}(\lambda) \geq 0$ and $\lambda \notin I_\sigma$.*

4. *$\text{Re}(\lambda) < 0$ and $\lambda \notin I_\mathfrak{a}$.*

Let $U \subset \partial X$ be a non-empty open subset, and let $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda))$ be such that $\phi_{|U} = 0$ and $(\hat{J}_{\sigma,\lambda}\phi)_{|U} = 0$. Then $\phi = 0$.

Before turning to the proof we recall that Lemma 5.3 implies that $(\hat{J}_{\sigma,\lambda}\phi)_{|U}$ is well-defined even if $\hat{J}_{\sigma,\lambda}$ has a pole.

Proof. We modify an argument given by van den Ban-Schlichtkrull [6] for the case $\sigma = 1$.

Assume that (σ, λ) satisfies condition 4. Since $\mathrm{Re}(\lambda) < 0$, the operator $\hat{J}_{\sigma,\lambda}$ is regular and non-vanishing. Since $\lambda \notin I_{\mathfrak{a}} \supset I_{\sigma}$ the principal series representation $\pi^{\sigma,\lambda}$ is irreducible. This implies that $\hat{J}_{\sigma,\lambda}$ is bijective. Moreover, $\hat{J}_{\sigma,-\lambda}$ is regular, since $-\lambda \notin I_{\mathfrak{a}}$. Thus, by the functional equation (23) we can reduce the proof to case 1, 2 or 3 replacing ϕ by $\hat{J}_{\sigma,\lambda}(\phi)$.

We now assume that $\mathrm{Re}(\lambda) \geq 0$, $\sigma = 1$ and $\phi \not\equiv 0$. Then the Poisson transform $P_{\lambda}\phi \in C^{\infty}(X)$ does not vanish and is annihilated by $D := \Delta_X - \rho^2 + \lambda^2$. Without loss of generality we can assume that $M \in U$. Since $P_{\lambda}\phi$ is real analytic and not identically zero the expansion (37) has non-trivial terms. Let m be the smallest integer such that $\psi_m \not\equiv 0$ near M, where we set $\psi_0 := \hat{J}_{1,\lambda}\phi$. We have to show that $m = 0$.

With respect to the coordinates k, a the operator D has the form $D = D_0 + a^{-2\alpha}R(a,k)$, where D_0 is a constant coefficient operator on A and R is a differential operator with coefficients which remain bounded if $a \to \infty$ (see [28], Ch. IV, §5, (8)). Moreover, it is known that D_0 coincides with the $\bar{N}$-radial part of D.

We consider the $\bar{N}$-invariant function $f \in C^{\infty}(X)$ defined by $f(\bar{n}a) := a^{-(\lambda+\rho+m\alpha)}$. Since D annihilates the asymptotic expansion (37) we have $Df = D_0f = 0$. On the other hand, f satisfies $(\Delta_X - \rho^2 + (\lambda + m\alpha)^2)f = 0$. Hence $(\lambda + m\alpha)^2 = \lambda^2$. Since $\mathrm{Re}(\lambda) \geq 0$ we conclude that $m = 0$.

The proof for general σ proceeds similarly. Let $\mathrm{Re}(\lambda) \geq 0$ and $\lambda \notin I_{\sigma}$. In particular, the principal series representation $\pi^{\sigma,\lambda}$ is irreducible. We choose $0 \neq T \in \mathrm{Hom}_M(V_{\sigma}, V_{\gamma})$ for a suitable $\gamma \in \hat{K}$. Then $P := P_{\lambda}^T$ is injective, and the range of P can be identified with the kernel of a certain invariant differential operator $D : C^{\infty}(X, V(\gamma)) \to C^{\infty}(X, V(\gamma'))$ for some representation γ' of K (see [10], Sec.3).

We now assume $\phi \not\equiv 0$. Moreover, without loss of generality we can assume that $M \in U$. Since $P\phi$ is real analytic, the expansion (37) does not vanish. Let m be the smallest integer such that $\psi_m \not\equiv 0$ near M (where $\psi_0 := T^w\hat{J}_{\sigma,\lambda}\phi$). Again, $D = D_0 + a^{-\alpha}R(a,k)$, where D_0 is the constant coefficient operator on A given by the $\bar{N}$-radial part of D, and R remains bounded if $a \to \infty$ (see [56], Thm. 9.1.2.4).

Choose $k \in K$ near 1 and $\sigma' \subset \gamma_{|M}$ such that that there exists an orthogonal projection $S \in \mathrm{Hom}_M(V_{\gamma}, V_{\sigma'})$ with $S\gamma(k)\psi_m(k) =: v \neq 0$. Consider the $\bar{N}$-invariant section $f \in C^{\infty}(X, V(\gamma))$ defined by

$$f(\bar{n}a) := a^{-(\lambda+\rho+m\alpha_1)}S^*v \ .$$

Since D annihilates the asymptotic expansion (37), one can check that $Df = D_0f = 0$ and thus $f = P\phi_1$ for some $\bar{N}$-invariant $\phi_1 \in C^{-\infty}(\partial X, V(\sigma_{\lambda}))$.

Now $f = P_{\lambda+m\alpha_1}^S\delta v$, where $\delta v \in C^{-\infty}(\partial X, V(\sigma'_{\lambda+m\alpha_1}))$ is the delta distribution at 1 with vector part v. Since D and $P_{\lambda+m\alpha_1}^S$ are G-equivariant and δv generates the G-module $C^{-\infty}(\partial X, V(\sigma'_{\lambda+m\alpha}))$, we obtain a non-trivial intertwining operator I from $C^{-\infty}(\partial X, V(\sigma'_{\lambda+m\alpha}))$ to the kernel of D, hence to $C^{-\infty}(\partial X, V(\sigma_{\lambda}))$. This implies

$$\chi_{\mu_{\sigma}+\rho_m-\lambda} = \chi_{\mu_{\sigma'}+\rho_m-\lambda-m\alpha} \ . \tag{44}$$

We conclude that $\chi_{\mu_{\sigma'}+\rho_m-\lambda-m\alpha}$ is not integral and thus $\pi^{\sigma',\lambda+m\alpha}$ is irreducible. Hence I is an isomorphism. Counting K-types one finds that $\sigma = \sigma'$ or $\sigma = w\sigma'$. It follows that $|\mu_\sigma + \rho_m|^2 = |\mu_{\sigma'} + \rho_m|^2$, and hence $\lambda^2 = (\lambda + m\alpha)^2$. The condition $\mathrm{Re}(\lambda) \geq 0$ implies that $m = 0$. Hence $(\hat{J}_{\sigma,\lambda}\phi)_{|U} \neq 0$.

In case $G = SL(2,\mathbb{R})$ and $\sigma = -1$ we argue as above in order to obtain (44) which is in this case equivalent to $\lambda^2 = (\lambda + m\alpha)^2$. This completes the proof of the lemma. $\qquad\square$

The above argument can be extended to cover also some cases of $\sigma \neq 1$, $\mathrm{Re}(\lambda) \geq 0$ with $\lambda \in I_\sigma$. This would lead to stronger vanishing results for $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ below. But there exist examples of $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}^*$ with $\mathrm{Re}(\lambda) \geq 0$ and $\lambda \in I_\sigma$, where the assertion of Lemma 6.7 is false. This is connected with the non-triviality of the spaces $U_\Lambda(\sigma_\lambda, \varphi)$ introduced in Definition 7.5 below.

Corollary 6.8 *Under the assumptions of Lemma 6.7*

$$res_\Omega \circ \hat{J}_{\sigma,\lambda} : C^{-\infty}(\Lambda, V(\sigma_\lambda)) \to C^\infty(\Omega, V(\sigma_{-\lambda}))$$

is injective. $\qquad\square$

For the remainder of this section we assume $X \neq \mathbb{O}H^2$. The following corollary now gives the first part of Theorem 6.1.

Corollary 6.9 *For fixed σ and φ the set $\{\lambda \in \mathfrak{a}^*_\mathbb{C} \,|\, ^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \neq 0\}$ is discrete.*

Proof. Assume that $ext : C^{-\infty}(B, V_B(\tilde{\sigma}_\lambda, \tilde{\varphi})) \to {}^\Gamma C^{-\infty}(\partial X, V(\tilde{\sigma}_\lambda, \tilde{\varphi}))$ is regular at λ. Then $res : {}^\Gamma C^{-\infty}(\partial X, V(\tilde{\sigma}_\lambda, \tilde{\varphi})) \to C^{-\infty}(B, V_B(\tilde{\sigma}_\lambda, \tilde{\varphi}))$ is surjective. It follows from Proposition 6.5 that $res \circ \hat{J}_\lambda \left({}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \right) = 0$. Assume in addition that $\lambda \notin I_\mathfrak{a}$. Then $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$ by Corollary 6.8. We conclude that $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \neq 0$ implies that ext is singular at λ or $\lambda \in I_\mathfrak{a}$. $\qquad\square$

We denote by $\mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_\cdot, \varphi))$ and $\mathcal{O}_\lambda C^{-\infty}(B, V_B(\sigma_\cdot, \varphi))$ the spaces of germs at λ of holomorphic families $\mu \mapsto f_\mu \in C^{-\infty}(\partial X, V(\sigma_\mu, \varphi))$ and $\mu \mapsto f_\mu \in C^{-\infty}(B, V_B(\sigma_\mu, \varphi))$, respectively.

Definition 6.10 *Let*

$$\mathcal{O}^0_\lambda C^{-\infty}(B, V_B(\sigma_\cdot, \varphi)) := \{f_\mu \in \mathcal{O}_\lambda C^{-\infty}(B, V_B(\sigma_\cdot, \varphi)) \,|\, (\mu - \lambda)ext f_\mu \in \mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_\cdot, \varphi))\} \,.$$

We define the space of invariant distributions on the limit set which are generated by the singular parts of ext by

$$E_\Lambda(\sigma_\lambda, \varphi) := \{res_{\mu=\lambda}ext(f_\mu) \,|\, f_\mu \in \mathcal{O}^0_\lambda C^{-\infty}(B, V_B(\sigma_\cdot, \varphi))\} \,.$$

Proposition 6.11 *The space $E_\Lambda(\sigma_\lambda, \varphi)$ is finite-dimensional and*

$$E_\Lambda(\sigma_\lambda, \varphi) \subset {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \ . \tag{45}$$

Assume that (σ, λ) satisfies the assumptions of Lemma 6.7. Then equality holds in (45) and ext is regular at λ if and only if ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$.

Proof.   Since *ext* has at most finite-dimensional singularities $E_\Lambda(\sigma_\lambda, \varphi)$ is finite-dimensional. (45) follows from the meromorphy of *ext* and the equation $res \circ ext = $ id. It remains to show that under the assumptions of Lemma 6.7

$$\dim E_\Lambda(\sigma_\lambda, \varphi) = \dim {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \ .$$

The main step in the proof is to show that $\dim E_\Lambda(\sigma_\lambda, \varphi) \geq \dim \mathrm{coker}(res)$ (see Formula (47) below). This is more or less obvious if *ext* has a pole of first order at λ. The general case is more involved.

Let L_λ be the multiplication operator on $\mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_., \varphi))$ and $\mathcal{O}_\lambda C^{-\infty}(B, V_B(\sigma_., \varphi))$ given by $(L_\lambda f)_\mu := (\mu - \lambda)f_\mu$. $\mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_., \varphi))$ becomes a Γ-module by $(\pi(g)f)_\mu := (\pi^{\sigma,\mu} \otimes \varphi)(g)f_\mu$, $g \in \Gamma$. The Γ-action commutes with L_λ. The restriction map induces a morphism of $\mathbb{C}[L_\lambda]$-modules

$$\mathcal{O}_\lambda res : {}^\Gamma \mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_., \varphi)) \to \mathcal{O}_\lambda C^{-\infty}(B, V_B(\sigma_., \varphi)) \ .$$

By Corollary 6.9 and Lemma 4.6 we have for $f \in {}^\Gamma \mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_., \varphi))$

$$ext \circ \mathcal{O}_\lambda res(f) = f \ . \tag{46}$$

Let k be the order of the pole of *ext* at λ. Set

$$\mathcal{O}_{\lambda,k} C^{-\infty}(\partial X, V(\sigma_., \varphi)) := \mathrm{coker}\left(L_\lambda^k : \mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_., \varphi)) \to \mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_., \varphi))\right) \ .$$

Then we consider the singular part of *ext* as an operator between $\mathbb{C}[L_\lambda]$-modules

$$ext^{<0} : \mathrm{coker}_{\mathcal{O}_\lambda} res \to {}^\Gamma \mathcal{O}_{\lambda,k} C^{-\infty}(\partial X, V(\sigma_., \varphi))$$

given by
$$ext^{<0}([f]) := ext(L_\lambda^k f) \bmod \mathrm{im} L_\lambda^k \ , \quad f \in \mathcal{O}_\lambda C^{-\infty}(B, V_B(\sigma_., \varphi)) \ ,$$

which is well-defined by (46). Assume that $ext^{<0}([f]) = 0$. Then $ext(L_\lambda^k f) = L_\lambda^k g$ for some $g \in \mathcal{O}_\lambda C^{-\infty}(\partial X, V(\sigma_., \varphi))$. It follows that $f = \mathcal{O}_\lambda res(g)$. We conclude that $ext^{<0}$ is injective. In particular, since $ext^{<0}$ is a finite-dimensional operator, the space $\mathrm{coker}_{\mathcal{O}_\lambda} res$ is finite-dimensional. The map

$$E_\Lambda(\sigma_\lambda, \varphi) \ni \mathrm{res}_{\mu=\lambda} ext(f_\mu) \mapsto ext^{<0}([f]) \in {}^\Gamma \mathcal{O}_{\lambda,k} C^{-\infty}(\partial X, V(\sigma_., \varphi)), \ f \in \mathcal{O}_\lambda^0 C^{-\infty}(B, V_B(\sigma_., \varphi)),$$

is well-defined and identifies $E_\Lambda(\sigma_\lambda, \varphi)$ with $\ker\left(L_\lambda : \mathrm{im}\, ext^{<0} \to \mathrm{im}\, ext^{<0}\right)$. We also consider the usual point-wise restriction map

$$res : {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$$

and the surjection

$$\mathrm{coker}\,(L_\lambda : \mathrm{coker}_{\mathcal{O}_\lambda} res \to \mathrm{coker}_{\mathcal{O}_\lambda} res) \to \mathrm{coker}(res)$$

induced by the point evaluation at λ. Summarizing the above discussion we obtain

$$
\begin{aligned}
d := \dim E_\Lambda(\sigma_\lambda, \varphi) &= \dim \ker \left(L_\lambda : \mathrm{im}\, ext^{<0} \to \mathrm{im}\, ext^{<0}\right) \\
&= \dim \mathrm{coker} \left(L_\lambda : \mathrm{im}\, ext^{<0} \to \mathrm{im}\, ext^{<0}\right) \\
&= \dim \mathrm{coker} \left(L_\lambda : \mathrm{coker}_{\mathcal{O}_\lambda} res \to \mathrm{coker}_{\mathcal{O}_\lambda} res\right) \\
&\geq \dim \mathrm{coker}(res) \, .
\end{aligned}
\tag{47}
$$

Set $\tilde{d} := \dim E_\Lambda(\tilde{\sigma}_\lambda, \tilde{\varphi})$. If (σ, λ) satisfies the assumptions of Lemma 6.7, then so does $(\tilde{\sigma}, \lambda)$. Thus Proposition 6.5 combined with Corollary 6.8 implies

$$\dim \mathrm{coker}(res) \geq \dim {}^\Gamma C^{-\infty}(\Lambda, V(\tilde{\sigma}_\lambda, \tilde{\varphi})) \, .$$

It eventually follows from (45) that

$$d \geq \dim\ \mathrm{coker}(res) \geq \dim {}^\Gamma C^{-\infty}(\Lambda, V(\tilde{\sigma}_\lambda, \tilde{\varphi})) \geq \tilde{d} \, .$$

Changing the roles of (σ, φ) and $(\tilde{\sigma}, \tilde{\varphi})$ we obtain $d = \tilde{d} = \dim {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. This finishes the proof of the proposition. $\qquad\square$

Corollary 6.12 *For every datum $(\sigma_\lambda, \varphi)$ the space ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ is finite-dimensional.*

Proof. Choose $\mu \in \mathfrak{a}^*$ large enough such that the finite-dimensional representation $\pi_{\sigma,\mu}$ exists and $\mathrm{Re}(\lambda + \mu) \geq 0$. Recall the definition (34) of the embedding $i_{\sigma,\mu}$. Since $(1, \lambda + \mu)$ satisfies the assumptions of Lemma 6.7 we obtain by Proposition 6.11 that

$$i_{\sigma,\mu} \left({}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))\right) \subset {}^\Gamma C^{-\infty}(\Lambda, V(1_{\lambda+\mu}, \varphi \otimes \pi_{\sigma,\mu})) = E_\Lambda(1_{\lambda+\mu}, \varphi \otimes \pi_{\sigma,\mu}) \, .$$

The corollary now follows since the space on the right hand side is finite-dimensional. $\qquad\square$

Thus we have completed the proof of Theorem 6.1. It is perhaps worth noting that the finite-dimensionality of ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ can also be proved without refering to the meromorphy of ext. In fact, using the asymptotic expansion (37) it is not difficult to show that there exists a number $k \in \mathbb{N}_0$ (depending on λ) and a continuous embedding

$$i : {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \hookrightarrow \left(C^k(\partial X, V(\tilde{\sigma}_{-\lambda}, \tilde{\varphi}))\right)' \, .$$

Furthermore, the Banach space topology on ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ induced by i coincides with the topology as a closed subspace of the Montel space $C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. It follows that the unit ball in the Banach space ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ is compact. This implies Corollary 6.12. However, arguments of this type seem not to be sufficient in order to establish results like Corollary 6.9.

7   Consequences of unitarity

From now on we assume that the finite-dimensional representation (φ, V_φ) of Γ is unitary. Then $\delta_\varphi = 0$. If $X = \mathbb{O}H^2$, then in addition we assume that $\delta_\Gamma < 0$. Hence in any case ext is meromorphic on a half-plane $\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda) > -\epsilon\}$ for some $\epsilon > 0$. The aim of this section is to work out the consequences of the unitarity of φ for the singularities of ext in that region. In particular, for fixed σ and φ we show that the set of parameters λ with $\mathrm{Re}(\lambda) \geq 0$ and allowing non-trivial invariant distributions with support on the limit set is finite and real. This result will be the main ingredient in the proof of the finiteness of the discrete spectrum of $\mathcal{Z}$ on $L^2(Y, V_Y(\gamma, \varphi))$, where $V_Y(\gamma, \varphi)$ is the hermitian vector bundle over Y defined by $V_Y(\gamma, \varphi) := \Gamma \backslash V(\gamma, \varphi)$.

In contrast to bilinear pairings $\langle ., . \rangle$ sesquilinear pairings will always be written as $(., .)$. By convention these pairings are $\mathbb{C}$-linear with respect to the first variable.

Since σ and φ are unitary representations, for $\lambda \in i\mathfrak{a}^*$ we have a positive definite conjugate linear pairing $(V_{\sigma_\lambda} \otimes V_\varphi) \otimes (V_{\sigma_\lambda} \otimes V_\varphi) \to V_{1-\rho}$. Since $V_B(1_{-\rho}) \cong \Lambda^{max} T^* B$ integration gives a natural L^2-scalar product on $C^\infty(B, V_B(\sigma_\lambda, \varphi))$. Let $L^2(B, V_B(\sigma_\lambda, \varphi))$ be associated Hilbert space. Using Lemma 5.8 we see that the adjoint S_λ^* with respect to this Hilbert space structure is just $S_{\bar{\lambda}} = S_{-\lambda}$.

Lemma 7.1 *If* $\mathrm{Re}(\lambda) = 0$, *then* S_λ *is regular and unitary.*

Proof.   Let λ be imaginary such that $S_{\pm\lambda}$ are regular. Let $f \in C^\infty(B, V_B(\sigma_\lambda, \varphi))$. Then by the functional equation (30)

$$\|S_\lambda f\|^2_{L^2(B, V_B(\sigma_\lambda, \varphi))} = (S_{-\lambda} \circ S_\lambda f, f)_{L^2(B, V_B(\sigma_\lambda, \varphi))} = \|f\|^2_{L^2(B, V_B(\sigma_\lambda, \varphi))} \ .$$

By meromorphy of S_λ this equation implies that S_λ is regular and unitary on $i\mathfrak{a}^*$. $\square$

Lemma 7.2 *If* $\mathrm{Re}(\lambda) = 0$ *and* $\lambda \neq 0$, *then*

$$ext : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$$

is regular and ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$.

Proof.   Note that (σ, λ) satisfies the assumptions of Lemma 6.7. Assume that ext is singular at a non-zero imaginary λ. The leading singular part of ext maps to distributions which are supported on the limit set Λ. Thus by Corollary 6.8 the scattering matrix $\hat{S}_\lambda = res \circ \hat{J}_\lambda \circ ext$ is singular at λ, too. Since $\hat{S}_\lambda = S_\lambda c_\sigma(-\lambda)$, and c_σ is regular on $i\mathfrak{a}^* \setminus \{0\}$, this contradicts Lemma 7.1. Thus ext is regular at λ. It follows by Proposition 6.11 that ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$. $\square$

Proposition 7.3

1. *If* $\mathrm{Re}(\lambda) > 0$, *then for any finite-dimensional representation* γ *of* K *and* $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ *the Poisson transform* P_λ^T *maps* ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ *to* $L^2(Y, V_Y(\gamma, \varphi))$. *If* ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \neq 0$, *then* λ *is real*.

2. *If* $ext : C^{-\infty}(B, V_B(\sigma_\mu, \varphi)) \to {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\mu, \varphi))$ *is singular at* $\mu = \lambda$, $\mathrm{Re}(\lambda) > 0$, *then* λ *is real and the order of the pole is* 1.

3. *If* $p_\sigma(0) \neq 0$, *i.e., the intertwining operator* $\hat{J}_0$ *is regular (see (23)), then* ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$ *for* $\mathrm{Re}(\lambda) > 0$, *and ext is regular in that region*.

Proof. Let γ be a finite-dimensional representation of K and $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$. Let $\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. By Lemma 6.2 for any compact subset $F \subset \Omega$ there exists a constant C such that for $a \in A_+$, $k \in FM$

$$|P_\lambda^T \phi(ka)| \leq C a^{-\lambda - \rho} . \tag{48}$$

In particular, $P_\lambda^T \phi \in L^2(Y, V_Y(\gamma, \varphi))$. This shows the first part of assertion 1.

Now let $\gamma \in \hat{K}$ be a minimal K-type of $\pi^{\sigma, \lambda}$ and T be injective. Then P_λ^T is injective. There is a real constant $c(\sigma)$ (see [11]) such that $(-\Omega_G + c(\sigma) + \lambda^2) \circ P_\lambda^T = 0$.

If $\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$, then $P_\lambda^T \phi$ is a square-integrable eigenfunction of $-\Omega_G + c(\sigma) + \lambda^2$ on Y. Since Y is complete Ω_G is essentially selfadjoint on the domain $C_c^\infty(Y, V_Y(\gamma, \varphi))$. Its selfadjoint closure has the domain of definition $\{f \in L^2(Y, V_Y(\gamma, \varphi)) \mid \Omega_G f \in L^2(Y, V_Y(\gamma, \varphi))\}$. In particular, Ω_G can not have non-trivial eigenvectors in $L^2(Y, V_Y(\gamma, \varphi))$ to eigenvalues with non-trivial imaginary part. Since $\mathrm{Re}(\lambda) > 0$, λ has to be real. This proves the second part of assertion 1.

Assume in addition that $p_\sigma(0) \neq 0$. Then there is a representation γ^s of K, a K-equivariant embedding $i : \gamma \hookrightarrow \gamma^s$ and a locally invariant Dirac operator $D(\sigma)$ acting on $C_c^\infty(Y, V_Y(\gamma^s, \varphi))$ such that

$$D(\sigma)^2 = -\Omega_G + c(\sigma) . \tag{49}$$

(If X is e.g. an odd-dimensional hyperbolic space , then $\sigma = \sigma' \oplus \sigma'^w$ for some non Weyl-invariant $\sigma' \in \hat{M}$ and γ_s is constructed in [11], see pp. 28-29 for (49).) Since $D(\sigma)$ is selfadjoint $-\Omega_G + c(\sigma)$ is non-negative. It follows that $\mathrm{Re}(\lambda) = 0$ which contradicts our assumption. Alternatively, one could use the fact ([3], Thm. 6.1) that in case $p_\sigma(0) \neq 0$ for $\mathrm{Re}(\lambda) > 0$ the principal series $\pi^{\sigma, \lambda}$ does not contain unitary subrepresentations. Compare Proposition 9.2 below. This proves assertion 3.

Now let $f_\mu \in C^{-\infty}(B, V_B(\sigma_\mu, \varphi))$, $\mu \in \mathfrak{a}_\mathbb{C}^*$, be a holomorphic family such that $ext(f_\mu)$ has a pole of order $k \geq 1$ at $\mu = \lambda$, $\mathrm{Re}(\lambda) > 0$. Let $0 \neq \phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ be the leading singular part of $ext(f_\mu)$ at $\mu = \lambda$. In particular, λ is real and $P_\lambda^T \phi \in L^2(Y, V_Y(\gamma, \varphi))$ by assertion 1.

Since the leading singular part of ext is a finite-dimensional operator we may assume that $f_\mu \in C^\infty(B, V_B(\sigma_\mu, \varphi))$ without changing ϕ. Then by Lemma 6.2 for any compact subset $F \subset \Omega$ and $\mu_0 > 0$ there exist constants C_1, C_2, C_3 such that for $a \in A_+$, $k \in FM$, $\mu_0 \le \mu < \lambda$,

$$
\begin{aligned}
|(\mu - \lambda)^k P_\mu^T ext(f_\mu)(ka)| &\le C_1 |\mu - \lambda|^k a^{\mu-\rho} + C_2 a^{-\mu-\rho} \\
&\le C_3 (1 + \log a)^{-k} a^{\lambda-\rho} .
\end{aligned}
\tag{50}
$$

There is a constant C_4 such that for $\mu_0 \le \mu < \lambda$

$$
|\mu - \lambda|^k |P_\mu^T ext(f_\mu)(x)| \, |P_\lambda^T \phi(x)| \le C_4 (1 + \log(a_x))^{-k} a_x^{-2\rho}, \quad x \in F .
$$

We now assume that $k \ge 2$. Then

$$
\int_{FK} (1 + \log(a_x))^{-k} a_x^{-2\rho} dx < \infty ,
$$

and we obtain by Lebesgue's Theorem of dominated convergence

$$
\begin{aligned}
\|P_\lambda^T \phi\|^2_{L^2(Y, V_Y(\gamma, \varphi))} &= (\lim_{\substack{\mu \to \lambda \\ \mu < \lambda}} (\mu - \lambda)^k P_\mu^T ext(f_\mu), P_\lambda^T \phi)_{L^2(Y, V_Y(\gamma, \varphi))} \\
&= \lim_{\substack{\mu \to \lambda \\ \mu < \lambda}} (\mu - \lambda)^k (P_\mu^T ext(f_\mu), P_\lambda^T \phi) .
\end{aligned}
$$

On the other hand the estimates (48) and (50) allow for partial integration, and we obtain for $\mu < \lambda$

$$
\begin{aligned}
(P_\mu^T ext(f_\mu), P_\lambda^T \phi) &= \frac{1}{\lambda^2 - \mu^2} ((-\Omega_G + c(\sigma) + \lambda^2) P_\mu^T ext(f_\mu), P_\lambda^T \phi) \\
&= \frac{1}{\lambda^2 - \mu^2} (P_\mu^T ext(f_\mu), (-\Omega_G + c(\sigma) + \lambda^2) P_\lambda^T \phi) \\
&= 0 .
\end{aligned}
$$

Hence $\|P_\lambda^T \phi\|_{L^2(Y, V_Y(\gamma, \varphi))} = 0$. Since P_λ^T is injective this contradicts $\phi \ne 0$. We conclude that $k = 1$. This proves assertion 2. $\square$

Proposition 7.4

1. *The order of the pole of ext at 0 is at most 1.*

2. *If $p_\sigma(0) \ne 0$, then ext is regular at 0.*

3. *If $p_\sigma(0) = 0$, then the residue $\mathrm{res}_{\mu=0} ext$ vanishes on the 1-eigenspace of the involution S_0 and identifies the -1-eigenspace with $E_\Lambda(\sigma_0, \varphi)$.*

Proof. The proof is analogous to the proof of Proposition 7.3. Let γ be the representation of K on the sum of the K-isotypic components of $C^\infty(\partial X, V(\sigma_0))$ that correspond to all minimal K-types of $C^\infty(\partial X, V(\sigma_0))$. ($\gamma$ is irreducible iff $p_\sigma(0) = 0$.) Then one can find $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ such that the Poisson transform

$$
P_\mu := P_\mu^T \otimes \mathrm{id} : C^{-\infty}(\partial X, V(\sigma_\mu, \varphi)) \to C^\infty(X, V(\gamma, \varphi))
$$

is injective for μ in a neighbourhood of 0.

Let $\mu \mapsto f_\mu \in C^\infty(B, V_B(\sigma_\mu, \varphi))$ be a holomorphic family defined on such a neighbourhood. Assume that $ext f_\mu$ has a pole of order k at 0. We want to study the leading singular part $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_0, \varphi))$ of $ext f_\mu$ at $\mu = 0$ via the leading singular part $P_0\phi$ of $P_\mu(ext f_\mu)$. For μ on a sufficiently small pointed disc D around 0 we obtain by Lemma 6.2, 3.,

$$(P_\mu(ext f_\mu))(ka) = a^{\mu-\rho}c_\gamma(\mu)Tf_\mu(k) + a^{-\mu-\rho}T^w(\hat{S}_\mu f_\mu)(k) + O(a^{-\rho-\epsilon}) \tag{51}$$

for $a \to \infty$ uniformly in μ and kM varying in compact subsets $D_0 \subset D$, $F \subset \Omega$.

If $p_\sigma(0) \neq 0$, then $c_\gamma(\mu)$ and $\hat{S}_\mu$ are regular at $\mu = 0$. We multiply (51) by μ^k and apply Cauchy's integral formula in order to conclude that for any compact subset $F \subset \Omega$ there exists a constant C such that for $a \in A_+$, $k \in FM$

$$|P_0\phi(ka)| \leq Ca^{-\rho-\epsilon} . \tag{52}$$

If $p_\sigma(0) = 0$, then $c_\gamma(\mu)$ and $\hat{S}_\mu$ have only first order poles at $\mu = 0$. If $k \geq 2$, then we can argue as above in order to obtain (52).

In particular, $P_0\phi \in L^2(Y, V_Y(\gamma, \varphi))$. By (51) and (52) the pairing $(P_\mu ext(f_\mu), P_0\phi)$ is defined for $0 < |\mu| < \epsilon$. As in the proof of Proposition 7.3 we show by partial integration that this pairing vanishes and that $\|P_0\phi\|^2_{L^2(Y, V_Y(\gamma, \varphi))} = 0$. Hence $\phi = 0$ unless $k = 1$ and $p_\sigma(0) = 0$.

It remains to show the last assertion of the proposition. Thus assume $k = 1$ and $p_\sigma(0) = 0$. Applying the residue theorem and (22) to (51) we obtain

$$(P_0\phi)(ka) = a^{-\rho}(\mathrm{res}_{\mu=0}c_\sigma(\mu))T^w(f_0(k) - S_0f_0(k)) + O(a^{-\rho-\epsilon}) .$$

Thus the leading asymptotic coefficient of $P_0\phi$ vanishes iff $S_0f_0 = f_0$. In this case $P_0\phi$ satisfies (52) which implies that $\phi = 0$ as above. We conclude that $\ker \mathrm{res}_{\mu=0}ext = \ker(S_0 - \mathrm{id})$. It follows from the definition of $E_\Lambda(\sigma_0, \varphi)$ that

$$\mathrm{res}_{\mu=0}ext : \ker(S_0 + \mathrm{id}) \to E_\Lambda(\sigma_0, \varphi)$$

is an isomorphism. The proof of the proposition is now complete. $\qquad\square$

For $0 \notin I_\sigma$ Proposition 7.4 could also have been proved by a refinement of the proof of Lemma 7.2.

Recall Definition 6.10 of $E_\Lambda(\sigma_\lambda, \varphi)$. For $\mathrm{Re}(\lambda) \geq 0$ we have just proved that this space coincides with the image of the residue of ext at λ.

Definition 7.5 *For all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we define the space of "stable" invariant distributions supported on the limit set by*

$$U_\Lambda(\sigma_\lambda, \varphi) := \{\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \mid res \circ \hat{J}_{\sigma,\lambda}(\phi) = 0\} .$$

Corollary 6.8 implies that if $\mathrm{Re}(\lambda) \geq 0$ and $U_\Lambda(\sigma_\lambda, \varphi)$ is non-trivial, then $\lambda \in I_\sigma$. We can now refine Proposition 6.11.

Proposition 7.6 *For* $\mathrm{Re}(\lambda) \geq 0$ *we have*

$$^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = E_\Lambda(\sigma_\lambda, \varphi) \oplus U_\Lambda(\sigma_\lambda, \varphi) .$$

Proof. We assume that $\lambda \in \mathfrak{a}^*$. Otherwise there is nothing to show. Let $\phi \in E_\Lambda(\sigma_\lambda, \varphi) \cap U_\Lambda(\sigma_\lambda, \varphi)$. Let P be an injective Poisson transform as in the proofs of Propositions 7.3 and 7.4. Using Lemma 6.2, 1. in case $\lambda = 0$ and 2. in case $\lambda > 0$, we see that there exists a constant $\epsilon > 0$ such that for any compact $F \subset \Omega$ there exists a constant C such that for $a \in A_+$, $k \in FM$

$$|P\phi(ka)| \leq C a^{-\lambda-\rho-\epsilon} .$$

As in the proof of Proposition 7.4 this implies $\phi = 0$. Thus $res \circ \hat{J}_\lambda$ is injective on $E_\Lambda(\sigma_\lambda, \varphi)$. It remains to show that

$$\dim E_\Lambda(\sigma_\lambda, \varphi) = \dim res \circ \hat{J}_\lambda \left(^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))\right) \tag{53}$$

By (47) we have $\dim E_\Lambda(\sigma_\lambda, \varphi) \geq \dim \mathrm{coker}(res)$. On the other hand Proposition 6.5 implies that

$$\dim \mathrm{coker}(res) \geq \dim res \circ \hat{J}_\lambda \left(^\Gamma C^{-\infty}(\Lambda, V(\tilde{\sigma}_\lambda, \tilde{\varphi}))\right) .$$

Since λ is real and φ is unitary the space on the right hand side is conjugate linear isomorphic to $res \circ \hat{J}_\lambda \left(^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))\right)$. This implies (53). $\square$

Definition 7.7 *Define*

$$\begin{aligned}
PS(\sigma, \varphi) &:= \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda) \geq 0,\ {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \neq 0\} , \\
PS_{\mathrm{res}}(\sigma, \varphi) &:= \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda) \geq 0,\ E_\Lambda(\sigma_\lambda, \varphi) \neq 0\} , \\
PS_U(\sigma, \varphi) &:= \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \mathrm{Re}(\lambda) \geq 0,\ U_\Lambda(\sigma_\lambda, \varphi) \neq 0\}.
\end{aligned}$$

Proposition 7.8 $PS(\sigma, \varphi) = PS_{\mathrm{res}}(\sigma, \varphi) \cup PS_U(\sigma, \varphi)$ *is a finite subset of the interval* $[0, \delta_\Gamma] \subset \mathfrak{a}^*$, *and we have* $PS_U(\sigma, \varphi) \subset I_\sigma$. *The space*

$$\bigoplus_{\lambda \in PS(\sigma, \varphi)} {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$$

is finite-dimensional. If $\lambda \in PS(\sigma, \varphi) \setminus I_\sigma$, *then* ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$ *is spanned by the residue of* ext *at* λ.

If $p_\sigma(0) \neq 0$, *then* $PS(\sigma, \varphi) = PS_U(\sigma, \varphi) \subset \{0\}$.

If $\delta_\Gamma > 0$, *then* $\delta_\Gamma \in PS_{\mathrm{res}}(1, 1)$ *and* $\dim {}^\Gamma C^{-\infty}(\Lambda, V(1_{\delta_\Gamma})) = 1$.

Proof. Combine the results of the present section with Theorem 6.1 and Theorem 4.7. The last assertion follows from the construction of the Patterson-Sullivan measure ([47], [52], [18]) and the uniqueness of the eigen*function* $f \in L^2(Y)$ corresponding to the smallest eigenvalue of Δ. □

Corollary 7.9 *If X is an odd-dimensional hyperbolic space and σ is a faithful representation of $M = Spin(n-1)$, then $PS(\sigma, \varphi)$ is empty.*

Proof. The condition on σ ensures that $0 \notin I_\sigma$ and that $p_\sigma(0) \neq 0$ ([32], Ch.12). □

8 Abstract harmonic analysis on $\Gamma \backslash G$

Let (φ, V_φ) be a unitary representation of Γ. We consider the Hilbert space

$$L^2(\Gamma \backslash G, \varphi) := \{f : G \to V_\varphi \mid f(gx) = \varphi(g)f(x) \; \forall g \in \Gamma, \; x \in G, \; \int_{\Gamma \backslash G} |f(x)|^2 \, dx < \infty\} \, .$$

The action of G on $\Gamma \backslash G$ by right translations induces a unitary representation of G on $L^2(\Gamma \backslash G, \varphi)$. The abstract Plancherel theorem (see e.g. [55], Thm. 14.10.5) provides a direct integral decomposition of this representation into irreducibles

$$L^2(\Gamma \backslash G, \varphi) \overset{\alpha}{\cong} \int_{\hat{G}}^{\oplus} M_\pi \hat{\otimes} V_\pi \, d\kappa(\pi) \, . \tag{54}$$

Here $\hat{G}$ denotes the unitary dual of G, V_π carries an irreducible unitary representation belonging to the class $\pi \in \hat{G}$, the Hilbert space M_π is called the multiplicity space of π in $L^2(\Gamma \backslash G, \varphi)$, and κ is a Borel measure on $\hat{G}$ called the Plancherel measure. The isomorphism (54) is a unitary equivalence of representations, where G acts on the right hand side by $\mathrm{id}_{M_\pi} \otimes \pi$. Throughout the paper we shall not distinguish between an element $\pi \in \hat{G}$ and a particular representative (π, V_π). Note that only the measure class of κ is uniquely defined. Strictly speaking, the spaces M_π are defined on a complement of a set of measure zero, only.

If Γ is the trivial group, then $M_\pi \cong V'_\pi$, and $M_\pi \hat{\otimes} V_\pi$ can be identified with the space $\mathcal{L}^2(V_\pi)$ of Hilbert-Schmidt operators on V_π which has a canonical scalar product. Then a canonical choice for α is $\alpha(f)(\pi) := \pi(f) \in \mathcal{L}^2(V_\pi)$, $f \in L^2(G)$, $\pi \in \hat{G}$. This determines the Plancherel measure. In contrast, for non-trivial Γ there is no obvious normalization of the unitary equivalence α, of the Hilbert space structure on $M_\pi \hat{\otimes} V_\pi$ and of the Plancherel measure.

The goal of this paper can now be phrased as follows: Make the equivalence (54) and all its ingredients as explicit as possible. In the present section we specify this task. In particular, following the approach of Bernstein [8] we identify M_π with a subspace of the "tempered"

functionals on the space of smooth vectors $V_{\pi,\infty}$ of V_π. This provides the bridge to the results of Section 7.

We introduce the Harish-Chandra Schwartz space $\mathcal{C}(G, V_\varphi)$ of G in a way suitable for our purposes. Fix a base $\{X_i\}$ of $\mathfrak{g}$ and let $\mathcal{I}_N$, $N \in \mathbb{N}_0$, denote the set of all multiindices $I = (i_1, \ldots, i_{\dim(\mathfrak{g})})$, $|I| \le N$. A multiindex $I \in \mathcal{I}_N$ defines an element $X_I = X_1^{i_1} \ldots X_{\dim(\mathfrak{g})}^{i_{\dim(\mathfrak{g})}} \in \mathcal{U}(\mathfrak{g})$. For $N \in \mathbb{N}_0$ and a V_φ-valued smooth function f on G we set

$$q_N(f)^2 := \sum_{(I,J) \in \mathcal{I}_N \times \mathcal{I}_N} \int_G |\log(a_g)^N f(X_I g X_J)|^2 \, dg \ .$$

Then $\mathcal{C}(G, V_\varphi) = \{f \mid q_N(f) < \infty \ \forall N \in \mathbb{N}_0\}$. The seminorms q_N, $N \in \mathbb{N}_0$ induce the structure of a Fréchet space on $\mathcal{C}(G, V_\varphi)$. For fixed $N \in \mathbb{N}_0$ we define the Hilbert space $\mathcal{C}^N(G, V_\varphi)$ as the completion of $\mathcal{C}(G, V_\varphi)$ with respect to q_N. Then we have continuous inclusions $\mathcal{C}(G, V_\varphi) \subset \mathcal{C}^N(G, V_\varphi) \subset \mathcal{C}^0(G, V_\varphi) = L^2(G, V_\varphi)$.

Let χ be the cut-off function χ constructed in Lemma 6.4. We consider χ as a right K-invariant function on G.

Definition 8.1 *We define the Schwartz space on $\Gamma\backslash G$ by*

$$\mathcal{C}(\Gamma\backslash G, \varphi) := \{f \in L^2(\Gamma\backslash G, \varphi) \mid \chi f \in \mathcal{C}(G, V_\varphi)\} \ .$$

It inherits the structure of a Fréchet space from $\mathcal{C}(G, V_\varphi)$. Similarly, we define intermediate Hilbert spaces $\mathcal{C}^N(\Gamma\backslash G, \varphi)$, $N \in \mathbb{N}_0$, by

$$\mathcal{C}^N(\Gamma\backslash G, \varphi) := \{f \in L^2(\Gamma\backslash G, \varphi) \mid \chi f \in \mathcal{C}^N(G, V_\varphi)\} \ .$$

Define $\mathcal{C}^{-N}(\Gamma\backslash G, \varphi)$ to be the conjugate linear dual of $\mathcal{C}^N(\Gamma\backslash G, \varphi)$. The space of tempered distributions $\mathcal{C}'(\Gamma\backslash G, \varphi)$ is then by definition the conjugate linear dual of $\mathcal{C}(\Gamma\backslash G, \varphi)$.

Let (γ, V_γ) be a finite-dimensional unitary representation of K. Then $L^2(Y, V_Y(\gamma, \varphi)) \cong [L^2(\Gamma\backslash G, \varphi) \otimes V_\gamma]^K$. Using this isomorphism we define for $* \in \{\emptyset, N, -N, \prime\}$

$$\mathcal{C}^*(Y, V_Y(\gamma, \varphi)) := [\mathcal{C}^*(\Gamma\backslash G, \varphi) \otimes V_\gamma]^K \ .$$

Lemma 8.2 *If N is sufficiently large, then the inclusion*

$$i : \mathcal{C}^N(\Gamma\backslash G, \varphi) \hookrightarrow L^2(\Gamma\backslash G, \varphi)$$

is Hilbert-Schmidt.

Proof. Consider the inclusion $j : \mathcal{C}^N(G, V_\varphi) \hookrightarrow L^2(G, V_\varphi)$. It is Hilbert-Schmidt for N sufficiently large (see e.g. [8]). Let $\chi' \in C_c^\infty(X \cup \Omega)$ be a second cut-off function such that $\chi'\chi = \chi$.

Again, we consider χ' as a function on G. Then $p_{\chi'} : L^2(G, V_\varphi) \to L^2(\Gamma\backslash G, \varphi)$ defined by $p_{\chi'}(f)(x) := \sum_{g\in\Gamma} \varphi(g)\chi'(g^{-1}x)f(g^{-1}x)$ is continuous. Let $m_\chi : \mathcal{C}^N(\Gamma\backslash G, \varphi) \to \mathcal{C}^N(G, V_\varphi)$ be the operator induced by multiplication with χ. Now we can factorize i over the Hilbert-Schmidt operator j: $i = p_{\chi'} \circ j \circ m_\chi$. The lemma follows. $\qquad\square$

In the following we choose N suffiently large. Fix an operator α providing the unitary equivalence (54). By a theorem of Gelfand/Kostyuchenko (see [8]) Lemma 8.2 implies that the composition

$$\mathcal{C}^N(\Gamma\backslash G, \varphi) \hookrightarrow L^2(\Gamma\backslash G, \varphi) \xrightarrow{\alpha} \int_{\hat{G}}^{\oplus} M_\pi \hat{\otimes} V_\pi \, d\kappa(\pi)$$

is pointwise defined, i.e., there exists a collection of continuous maps

$$\alpha_\pi : \mathcal{C}^N(\Gamma\backslash G, \varphi) \to M_\pi \hat{\otimes} V_\pi, \quad \pi \in \hat{G}$$

such that for $f \in \mathcal{C}^N(\Gamma\backslash G, \varphi)$ we have $\alpha(f)(\pi) = \alpha_\pi(f)$. By changing α_π on a set of π's of measure zero (w.r.t. κ) we can assume that for all $\pi \in \hat{G}$ the map α_π is an intertwining operator of G-representations. Let

$$\beta_\pi : M_\pi \hat{\otimes} V_\pi \to \mathcal{C}^{-N}(\Gamma\backslash G, \varphi)$$

denote the adjoint of α_π. Note that $\mathcal{C}^{-N}(\Gamma\backslash G, \varphi) \subset \mathcal{C}'(\Gamma\backslash G, \varphi)$. The composition of β_π with this inclusion will also be denoted by β_π.

Let (π, V_π) be a representation of G on a reflexive Banach space, and let $(\pi', V_{\pi'})$ be its dual representation. The space of distribution vectors $V_{\pi,-\infty}$ of V_π is by definition $(V_{\pi',\infty})'$, where the subscript ∞ indicates the transition to the subspace of smooth vectors, and the second dualization is with respect to the canonical Fréchet topology on $V_{\pi',\infty}$. Then we have the following inclusions of G-representations: $V_{\pi,\infty} \subset V_\pi \subset V_{\pi,-\infty}$.

Definition 8.3 *Let (π, V_π) be a representation of G on a reflexive Banach space, and let (φ, V_φ) be a finite-dimensional unitary representation of Γ. An invariant distribution vector $\phi \in {}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)$ is called tempered (square integrable, resp.) if for all $v \in V_{\pi',\infty}$ the function*

$$G \ni g \mapsto c_{\phi,v}(g) := \langle \phi, \pi'(g)v \rangle \in V_\varphi$$

belongs to $\mathcal{C}'(\Gamma\backslash G, \varphi)$ $(L^2(\Gamma\backslash G, \varphi))$. By ${}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)_d \subset {}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)_{temp} \subset {}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)$ we denote the linear subspaces of square integrable and tempered invariant distribution vectors.

If (π, V_π) is an admissible representation of finite length, we have the following characterizations and consequences of temperedness.

Lemma 8.4 *Let (π, V_π) be an admissible G-representation of finite length on a reflexive Banach space, and let $V_{\pi',K} \subset V_{\pi',\infty}$ be the underlying $(\mathfrak{g}, K)$-module of K-finite elements of the dual representation $V_{\pi'}$. If $S \subset V_{\pi',K}$ is a generating set and $\phi \in {}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)$, then the following conditions are equivalent:*

1. $c_{\phi,v} \in \mathcal{C}'(\Gamma \backslash G, \varphi)$ *for all* $v \in S$.

2. $\phi \in {}^{\Gamma}(V_{\pi,-\infty} \otimes V_{\varphi})_{temp}$.

3. $c_{\phi,v} \in \mathcal{C}'(\Gamma \backslash G, \varphi)$ *for all* $v \in V_{\pi',\infty}$, *and the map* $c_{\phi,\cdot} : V_{\pi',\infty} \to \mathcal{C}'(\Gamma \backslash G, \varphi)$ *is continuous.*

The analogous assertions for ${}^{\Gamma}(V_{\pi,-\infty} \otimes V_{\varphi})_d$ *and* $L^2(\Gamma \backslash G, \varphi)$ *hold true, too.*

Proof. The lemma is a consequence of the globalization theory of Casselman and Wallach ([15], [55], Ch. 11). The conclusions 3. $\Rightarrow$ 2. $\Rightarrow$ 1. are obvious. We outline the argument for 1.$\Rightarrow$ 3.

Assume 1. Since the $(\mathfrak{g}, K)$-module $V_{\pi',K}$ is finitely generated and admissible, and K-finite matrix coefficients satisfy elliptic differential equations one can show that there exists $N \in \mathbb{N}_0$ such that $c_{\phi,v} \in \mathcal{C}^{-N}(\Gamma \backslash G, \varphi)_K$ for all $v \in V_{\pi',K}$. Here $\mathcal{C}^{-N}(\Gamma \backslash G, \varphi)_K$ denotes the subspace of K-finite smooth vectors of $\mathcal{C}^{-N}(\Gamma \backslash G, \varphi)$. But $\mathcal{C}^{-N}(\Gamma \backslash G, \varphi)$ is a Hilbert space on which G acts continuously. The theorem of Casselman and Wallach now states that the $(\mathfrak{g}, K)$-module homomorphism $V_{\pi',K} \ni v \mapsto c_{\phi,v} \in \mathcal{C}^{-N}(\Gamma \backslash G, \varphi)_K$ extends to a continuous homomorphism $V_{\pi',\infty} \to \mathcal{C}^{-N}(\Gamma \backslash G, \varphi)_\infty$. Condition 3 follows. The argument for ${}^{\Gamma}(V_{\pi,-\infty} \otimes V_{\varphi})_d$ is essentially the same. $\square$

The notion of temperedness is compatible with the notion of a tempered (irreducible) representation (π, V) of G as follows. For a moment let Γ and V_φ be trivial, and define $V_{-\infty,temp}$ as above. Then π is tempered iff $V_{-\infty,temp} \subset V_{-\infty}$ is dense. Similarly, π is square integrable, i.e., belongs to the discrete series, iff $V_{-\infty,d} \subset V_{-\infty}$ is dense. The above lemma in mind it is not difficult to see that this characterization of temperedness and square-integrability is equivalent to the various definitions appearing in the literature. In fact, more is true:

Lemma 8.5 *If* (π, V) *is an admissible* G-*representation of finite length on a reflexive Banach space, then* π *is tempered iff* $V_{-\infty,temp} = V_{-\infty}$. *Moreover, if* $v \in V'_\infty$ *is fixed, then the matrix coefficient map*

$$V_{-\infty} \ni \phi \mapsto c_{\phi,v} \in \mathcal{C}'(G)$$

is continuos.

Proof. The lemma follows from Lemma 10 in [2] which is based on subtle estimates of K-finite matrix coefficients. However, we would like to indicate a different argument which is more in the spirit of the proof of Lemma 8.4.

Fix $v \in V'_\infty$. Then for all $\phi \in V_K$ the asymptotic expansions of matrix coefficients ([54], 4.4.3.) give $c_{\phi,v} \in \mathcal{C}'(G)$. Thus by Lemma 8.4 we have $v \in V'_{-\infty,temp}$. As in the proof of Lemma 8.4 the map $\phi \mapsto c_{\phi,v}$ extends to a continuous map $c_{\cdot,v} : V_\infty \to \mathcal{C}^{-N}(G)_\infty$ for some $N \in \mathbb{N}_0$. Since $\mathcal{C}^{-N}(G)_\infty$ is a Hilbert space this map has a continuous right inverse, the adjoint of which is the continuous extension of $c_{\cdot,v}$ to $V_{-\infty}$

$$c_{\cdot,v} : V_{-\infty} \to \mathcal{C}^N(G)_{-\infty} \subset \mathcal{C}'(G) \ .$$

This proves the lemma. $\qquad\square$

We now return to the discussion of the map β_π.

Lemma 8.6 *For any $\pi \in \hat{G}$ there is an embedding*

$$i_\pi : M_\pi \hookrightarrow {}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_{temp}$$

such that for $m \in M_\pi$ and $v \in V_{\pi,\infty}$ we have $\beta_\pi(m \otimes v) = c_{i_\pi(m),v}$.

Proof. Let N be sufficiently large. Fix $m \in M_\pi$ and consider the G-intertwining operator $F_m : V_\pi \to \mathcal{C}^{-N}(\Gamma\backslash G, \varphi)$ given by $F_m(v) := \beta_\pi(m \otimes v)$. Then

$$F_m(V_{\pi,\infty}) \subset \mathcal{C}^{-N}(\Gamma\backslash G, \varphi)_\infty \subset \mathcal{C}^{-N}(\Gamma\backslash G, \varphi) \cap C^\infty(\Gamma\backslash G, \varphi) \ .$$

In particular, elements of $F_m(V_{\pi,\infty})$ can be evaluated at the identity $e \in G$. We define i_π by $\langle i_\pi(m), v \rangle := F_m(v)(e)$, $v \in V_{\pi,\infty}$. The assertion of the lemma is now obvious. $\qquad\square$

Our first concretization of the abstract Plancherel decomposition (54) is given by the following corollary.

Corollary 8.7 *There exists a collection of Hilbert spaces $N_\pi \subset {}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_{temp}$, $\pi \in \hat{G}$, and a direct integral*

$$\int_{\hat{G}}^\oplus N_\pi \hat{\otimes} V_\pi \, d\kappa(\pi)$$

such that the following holds:

1. *The matrix coefficient map $c : {}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_{temp} \otimes V_{\pi,\infty} \to \mathcal{C}'(\Gamma\backslash G, \varphi)$ gives rise to a map*

   $$c_\pi : N_\pi \hat{\otimes} V_\pi \to \mathcal{C}'(\Gamma\backslash G, \varphi) \ .$$

2. *Let $\mathcal{F}_\pi : \mathcal{C}(\Gamma\backslash G, \varphi) \to N_\pi \hat{\otimes} V_\pi$ be the adjoint of c_π. Then the collection of maps $\mathcal{F}_\pi$ extends to a unitary equivalence*

   $$\mathcal{F} : L^2(\Gamma\backslash G, \varphi) \overset{\cong}{\longrightarrow} \int_{\hat{G}}^\oplus N_\pi \hat{\otimes} V_\pi \, d\kappa(\pi) \ .$$

3. *If ${}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d \neq 0$, then $N_\pi = {}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d$.*

In particular, the Plancherel measure κ is supported on the set

$$\{\pi \in \hat{G} \,|\, {}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_{temp} \neq 0\}$$

and $\kappa(\{\pi\}) \neq 0$ iff $^{\Gamma}(V_{\pi',-\infty} \otimes V_\varphi)_d \neq 0$.

The Plancherel measure and the scalar product on the subspace $^{\Gamma}(V_{\pi',-\infty} \otimes V_\varphi)_d \subset N_\pi$ can be chosen such that $\kappa(\{\pi\}) = 1$, if $(V_{\pi',-\infty} \otimes V_\varphi)_d \neq 0$, and such that c_π induces an isometric embedding of $^{\Gamma}(V_{\pi',-\infty} \hat{\otimes} V_\varphi)_d \otimes V_\pi$ into $L^2(\Gamma\backslash G, \varphi)$.

Proof.   Set $N_\pi := i_\pi(M_\pi)$. $\square$

The plan of the rest of the paper is now as follows. In the next section we determine the spaces $(V_{\pi,-\infty} \otimes V_\varphi)_{temp}$ and $(V_{\pi,-\infty} \otimes V_\varphi)_d$ for all $\pi \in \hat{G}$. This is based on the results of the Section 7. In Section 10 we study wave packets of Eisenstein series. It turns out that they span the orthogonal complement to the discrete subspace

$$L^2(\Gamma\backslash G, \varphi)_d := \overset{\text{Hilbert}}{\underset{\{\pi\in\hat{G}\,|\,(V_{\pi,-\infty}\otimes V_\varphi)_d\neq 0\}}{\bigoplus}} \operatorname{im} c_\pi \subset L^2(\Gamma\backslash G, \varphi) \ .$$

The proof of this fact heavily depends on our a priori knowledge of the support of the Plancherel measure. The last section contains the summary of our results, including the determination of the scalar products on N_π and of the Placherel measure, as well as the consequences for the spectral theory of the Casimir operator acting on sections of the locally homogeneous vector bundle $V_Y(\gamma, \varphi)$ over the Kleinian manifold Y.

9   Tempered invariant distribution vectors

In this section we determine the tempered and square integrable invariant distribution vectors for all $\pi \in \hat{G}$.

First we need a rough classification of the unitary dual $\hat{G}$. Recall the notions of temperedness and square integrability of an irreducible representation (see the discussion following Lemma 8.4). The unitary dual is a disjoint union of the dicrete series, the unitary principal series, and the complementary series

$$\hat{G} = \hat{G}_d \cup \hat{G}_u \cup \hat{G}_c \ ,$$

where

$$
\begin{aligned}
\hat{G}_d &:= \{\pi \in \hat{G} \,|\, V_\pi \text{ is square integrable}\} \ , \\
\hat{G}_u &:= \{\pi \in \hat{G} \,|\, V_\pi \text{ is tempered}\} \setminus \hat{G}_d \ , \\
\hat{G}_c &:= \{\pi \in \hat{G} \,|\, V_\pi \text{ is not tempered}\} \ .
\end{aligned}
$$

The discrete series $\hat{G}_d$ has been determined by Harish-Chandra. It is empty iff $X = H^n$, n odd. In the other cases one can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is contained in $\mathfrak{k}$. An

infinitesimal character χ_λ, $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, is called regular if no expression of the form (43) vanishes. Let $W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the Weyl group of K. Then $\hat{G}_d$ can be parametrized by the Harish-Chandra parameters

$$\{\lambda \in \mathfrak{h}_{\mathbb{C}}^* \mid \chi_\lambda \text{ is regular and integral}\}/W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \tag{55}$$

such that $\mathcal{Z}$ acts on $\pi \in \hat{G}_d$ with infinitesimal character χ_λ, and π has the minimal K-type with highest weight $\lambda + \rho_\mathfrak{g} - 2\rho_\mathfrak{k}$ (see e.g. [54], Ch. 6 and Ch. 8). Here $\rho_\mathfrak{g}$ and $\rho_\mathfrak{k}$ are the half sums of the positive (w.r.t. λ) roots of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathfrak{k}$, respectively. In particular, for any $\gamma \in \hat{K}$ there are only finitely many discrete series representations containing the K-type γ. Strictly speaking, (55) parametrizes the discrete series representations for the linear group G with Lie algebra $\mathfrak{g}$ which has a simply connected complexification $G_{\mathbb{C}}$. In general, the parametrization remains valid, if one sharpens the notion of integrality.

The set $\hat{G}_u$ consists of the unitary principal series representations $\pi^{\sigma, \lambda}$, $\sigma \in \hat{M}$, $\mathrm{Re}(\lambda) = 0$. They are irreducible unless $\sigma = \sigma^w$, $p_\sigma(0) = 0$ and $\lambda = 0$. In the latter case we have $\pi^{\sigma, 0} = \pi^{\sigma, +} \oplus \pi^{\sigma, -}$, where the irreducible representations $\pi^{\sigma, \pm}$, called the non-degenerate limits of discrete series, are the ± 1-eigenspaces of $J_{\sigma, 0}$. All equivalences between these representations are induced by the intertwining operators $J_{\sigma, \lambda}$. For all that see [31], Ch. XIV.

Though also $\hat{G}_c$ is completely known (see [4] and the references therein) for our purposes less information is sufficient. The Langlands classification (see [54], Ch. 5) associates to any irreducible non-tempered representation (π, V_π) a unique Langlands parameter (σ, λ), $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\mathrm{Re}(\lambda) > 0$, such that $V_{\pi, \pm\infty}$ is equivalent to the unique irreducible subrepresentation of the principal series representation $\pi^{\sigma, \lambda}$ acting on $C^{\pm\infty}(\partial X, V(\sigma_\lambda))$. We denote this subrepresentation by $(\bar{\pi}^{\sigma, \lambda}, I_{\pm\infty}^{\sigma, \lambda})$. It is the image of $J_{\sigma^w, -\lambda}$. It is also the unique irreducible quotient of $C^\infty(\partial X, V(\sigma_{-\lambda}^w))$. If $\pi \in \hat{G}_c$, then $\sigma = \sigma^w$, $\lambda \in \mathfrak{a}^*$ (e.g. [31], Thm. 16.6.) and $p_\sigma(0) \neq 0$ ([3], Thm. 6.1). An invariant pre-Hilbert structure $(.,.)$ on $I_\infty^{\sigma, \lambda}$ realized as a quotient of $C^\infty(\partial X, V(\sigma_{-\lambda}))$ can now be described as follows: Let $(.,.)_0$ be the invariant sesquilinear pairing between $C^\infty(\partial X, V(\sigma_{-\lambda}))$ and $C^\infty(\partial X, V(\sigma_\lambda))$. Then

$$([f], [g]) := (f, J_{\sigma, -\lambda}(g))_0 , \tag{56}$$

where we have represented $[f], [g] \in I_\infty^{\sigma, \lambda}$ by $f, g \in C^\infty(\partial X, V(\sigma_{-\lambda}))$. Indeed, since ${}^t J_{\sigma, -\lambda} = J_{\sigma, -\lambda}$ the pairing (56) is hermitian. By $I^{\sigma, \lambda}$ we denote the Hilbert space completion of $I_\infty^{\sigma, \lambda}$ with respect to $(.,.)$.

We recall the relation between the Poisson transform P_λ^T and the matrix coefficients of principal series representations. For later reference we will state it as a lemma. Its verification is a standard computation with integral formulas.

Lemma 9.1 *Let φ, σ, γ be finite-dimensional representations of Γ, M, and K, respectively, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$. We consider the Poisson transform*

$$P_\lambda^T : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^\infty(X, V(\gamma, \varphi)) \cong [C^\infty(G, V_\varphi) \otimes V_\gamma]^K .$$

Then for any $v \in V_{\bar{\gamma}}$, $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$, $g \in G$ we have

$$\langle P_\lambda^T \phi(g), v \rangle = c_{\phi, v_T}(g) \in V_\varphi ,$$

where $v_T \in C^\infty(\partial X, V(\tilde{\sigma}_{-\lambda}))$ is the element defined by Frobenius reciprocity $v_T(k) := {}^t T \tilde{\gamma}(k^{-1})v$, $k \in K$. $\square$

Fix a finite-dimensional unitary representation (φ, V_φ) of Γ. Recall the decomposition ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = E_\Lambda(\sigma_\lambda, \varphi) \oplus U_\Lambda(\sigma_\lambda, \varphi)$ from Proposition 7.6.

Proposition 9.2 *Let $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\mathrm{Re}(\lambda) > 0$. Then*

$$ {}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_d = {}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_{temp} = {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = E_\Lambda(\sigma_\lambda, \varphi) \oplus U_\Lambda(\sigma_\lambda, \varphi) \ . $$

If one of these spaces is non-zero, then $\bar{\pi}^{\sigma,\lambda} \in \hat{G}_c$.

Proof.   Observe that ${}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_{temp} \subset {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_{temp}$. Fix a minimal K-type γ of $C^\infty(\partial X, V(\sigma_\lambda))$ together with a non-trivial $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$. We consider the injective Poisson transform

$$ P = P_\lambda^T : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^\infty(X, V(\gamma, \varphi)) \ . $$

Recall the definition of v_T from Lemma 9.1. If $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_{temp}$, then by definition $c_{\phi, v_T} \in \mathcal{C}'(\Gamma \backslash G, \varphi)$ for all $v \in V_{\tilde{\gamma}}$, hence $P\phi \in \mathcal{C}'(Y, V_Y(\gamma, \varphi))$.

Let $f \in C^\infty(\partial X, V((\tilde{\gamma}_{|M})_{-\lambda}, \tilde{\varphi}))$ with $\mathrm{supp}(f) \subset \Omega$. We want to show that $\langle \phi, {}^t T f \rangle = 0$. As in the proof of Theorem 4.7 we extend f to a section $\tilde{f} \in C^\infty(X, V(\tilde{\gamma}, \tilde{\varphi}))$. By Corollary 6.3 there is a constant C such that

$$ \langle \phi, {}^t T f \rangle = C \lim_{a \to \infty} a^{\rho - \lambda} \int_K \langle P\phi(ka), f(k) \rangle \ dk = C \lim_{n \to \infty} \int_G \langle P\phi(x), \tilde{f}_n(x) \rangle \ dx, $$

where $\tilde{f}_n(x) := a_x^{-(\lambda+\rho)} \psi(\log(a_x - n))\tilde{f}(x)$ for some $\psi \in C_c^\infty(0,1)$ satisfying $\int_0^1 \psi(t)dt = 1$. Define $F_n \in C_c^\infty(Y, V_Y(\tilde{\gamma}, \tilde{\varphi})) \subset \mathcal{C}(Y, V_Y(\tilde{\gamma}, \tilde{\varphi}))$ by

$$ F_n(x) := \sum_{g \in \Gamma} \tilde{\varphi}(g) \tilde{f}_n(g^{-1}x) \ . $$

We claim that $\lim_{n \to \infty} F_n = 0$ in $\mathcal{C}(Y, V_Y(\tilde{\gamma}, \tilde{\varphi}))$. Using that $\mathrm{supp}(f) \subset \Omega$ we find a finite subset $L \subset \Gamma$ such that

$$ \chi(x)F_n(x) = \chi(x) \sum_{g \in L} \tilde{\varphi}(g) \tilde{f}_n(g^{-1}x) = \sum_{g \in L} \tilde{\varphi}(g)((g^{-1})^* \chi \tilde{f}_n)(g^{-1}x) \ , $$

where χ is the cut-off function as in Lemma 6.4. We have $\lim_{n \to \infty} \tilde{f}_n = 0$ in $\mathcal{C}(X, V(\tilde{\gamma}, \tilde{\varphi}))$ and hence $(g^{-1})^* \chi \tilde{f}_n \to 0$ in $\mathcal{C}(X, V(\tilde{\gamma}, \tilde{\varphi}))$. This shows the claim.

Since $P\phi \in \mathcal{C}'(Y, V_Y(\gamma, \varphi))$ we obtain

$$ \langle \phi, {}^t T f \rangle = C \lim_{n \to \infty} \int_{\Gamma \backslash G} \langle P\phi(x), F_n(x) \rangle \ dx = C \lim_{n \to \infty} \langle P\phi, F_n \rangle = 0 \ . $$

This proves that ${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_{temp} \subset {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$.

On the other hand, we have by Proposition 7.3 that $P\left({}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))\right) \subset L^2(Y, V_Y(\gamma, \varphi))$. Since the elements v_T, $v \in V_{\tilde\gamma}$, generate the $(\mathfrak{g}, K)$-module $C^\infty(\partial X, V(\tilde\sigma_{-\lambda}))_K$ it follows from Lemma 8.4 and Lemma 9.1 that $c_{\phi,f} \in L^2(\Gamma\backslash G, \varphi)$ for all $f \in C^\infty(\partial X, V(\tilde\sigma_{-\lambda}))$. Thus ${}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \subset {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d$. It remains to show that ${}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d \subset {}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_d$. Indeed, let $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d$ and consider the G-map

$$c_{\phi,.} : C^\infty(\partial X, V(\tilde\sigma_{-\lambda})) \to L^2(\Gamma\backslash G, \varphi) .$$

Since the target space is a unitary representation of G the image of $c_{\phi,.}$ decomposes into a direct sum of irreducible representations. But $C^\infty(\partial X, V(\tilde\sigma_{-\lambda}))$ has the unique irreducible quotient

$$C^\infty(\partial X, V(\tilde\sigma_{-\lambda}))/(I_{-\infty}^{\sigma,\lambda})^\perp .$$

Thus $c_{\phi,.}$ factorizes over this quotient, and hence $\phi \in {}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_d$. Now, if ϕ is non-trival, then we can pull back the invariant pre-Hilbert structure from $L^2(\Gamma\backslash G, \varphi)$ to this quotient. By duality this induces an invariant scalar product on $I_\infty^{\sigma,\lambda}$. Hence $\bar\pi^{\sigma,\lambda} \in \hat G_c$. In view of the chain of inclusions

$$\begin{aligned}{}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_{temp} &\subset {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_{temp} \subset {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) \subset \\ &\subset {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d \subset {}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_d\end{aligned}$$

the proof of the proposition is now complete. □

Corollary 9.3 *The space ${}^\Gamma(I_{-\infty}^{\sigma,\lambda} \otimes V_\varphi)_d$ is finite-dimensional. It is non-trivial iff λ belongs to the finite set $PS(\sigma, \varphi) \setminus \{0\} \subset (0, \delta_\Gamma]$.*

Proof. Combine Proposition 9.2 with Proposition 7.8. □

The case $\sigma = 1$ is particularly interesting. In the following table we give the set of $\lambda > 0$ with $I^{1,\lambda} \in \hat G$.

X	$\mathbb{R}H^n$	$\mathbb{C}H^n$	$\mathbb{H}H^n$	$\mathbb{O}H^n$
λ	$(0, \rho]$	$(0, \rho]$	$(0, \rho - 2\alpha] \cup \{\rho\}$	$(0, \rho - 6\alpha] \cup \{\rho\}$

.

If $\delta_\Gamma > 0$, then by Proposition 9.2 the representation I^{1,δ_Γ} is unitary. This leads to the restriction of the set of possible values of δ_Γ found by Corlette [18] (see Section 2).

Lemma 9.4 *If $(\pi, V_\pi) \in \hat G$ is tempered, then ${}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)_{temp} = {}^\Gamma(V_{\pi-\infty} \otimes V_\varphi)$. Moreover, for any $v \in V_{\tilde\pi,\infty}$ the map*

$${}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi) \ni \phi \mapsto c_{\phi,v} \in \mathcal{C}'(\Gamma\backslash G, \varphi)$$

is continuous. If, in addition, $\pi = \pi^{\sigma,\lambda}$, $\lambda \neq 0$ imaginary, then ext identifies $C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ with ${}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)_{temp}$.

Proof. Observe that there is a natural inclusion $^\Gamma \mathcal{C}'(G, V_\varphi) \hookrightarrow \mathcal{C}'(\Gamma \backslash G, \varphi)$ induced by the adjoint of the multiplication by χ. The first assertions of the lemma now follow from Lemma 8.5. The last one follows from Lemmas 4.6 and 7.2. $\square$

Proposition 9.5 *Let* $\sigma \in \hat{M}$. *If* $\lambda \neq 0$ *is imaginary, then* $^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d = 0$. *For* $\lambda = 0$ *we have*

$$^\Gamma C^{-\infty}(\partial X, V(\sigma_0, \varphi))_d = U_\Lambda(\sigma_0, \varphi) .$$

If this finite-dimensional space is non-trivial, then $\sigma \neq 1$ *and* $0 \in I_\sigma$.

Proof. We claim that for $\mathrm{Re}(\lambda) = 0$

$$^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d \subset {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) . \tag{57}$$

We now prove the proposition assuming the claim. If $\lambda \neq 0$, then by Lemma 7.2 we have $^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi)) = 0$ and hence $C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d = 0$. Now consider the case $\lambda = 0$. If $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))_d$, then the claim implies $res_\Omega \phi = 0$. By the asymptotic expansion Lemma 6.2, 2., we see that ϕ is square-integrable iff $res_\Omega \circ \hat{J}_0 \phi = 0$. This proves that $^\Gamma C^{-\infty}(\partial X, V(\sigma_0, \varphi))_d = U_\Lambda(\sigma_0, \varphi)$. If the latter space is non-trivial, then $\sigma \neq 1$ and $0 \in I_\sigma$ by Corollary 6.8.

The proof of (57) is analogous to the proof of Proposition 9.2. Let $\phi \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. One has to show that for some Poisson transform P the condition $P\phi \in L^2(Y, V_Y(\gamma, \varphi))$ implies $res(\phi) = 0$. We give the argument for the most involved case that σ is Weyl-invariant, $\lambda = 0$, and $\hat{J}_0$ regular. The similar treatment of the remaining cases is left to the reader.

In fact, in this case we show at once that $res(\phi) = 0$ and $res(\hat{J}_0 \phi) = 0$. Recall that $\pi^{\sigma,0}$ splits into the ± 1-eigenspaces $\pi^{\sigma,\pm}$ of J_0. Let $\gamma_\pm$ be the minimal K-type of $\pi^{\sigma,\pm}$. Choose embeddings $T^\pm \in \mathrm{Hom}_M(V_\sigma, V_{\gamma_\pm})$ and set $P^\pm := P_0^{T^\pm} \otimes \mathrm{id}$. Define $t^\pm \in \mathrm{Hom}_M(V_{\tilde\sigma}, V_{\tilde\gamma_\pm})$ by $^t(T^\pm)^w \circ t^\pm = \mathrm{id}_{V_{\tilde\sigma}}$. Using (36) and Corollary 6.3 we have for $f \in C^\infty(\partial X, V(\tilde\sigma_0, \tilde\varphi))$

$$\langle c_{\gamma_\pm}(0)T^\pm \phi + (T^\pm)^w \hat{J}_0 \phi, t^\pm f \rangle = \lim_{a\to\infty} a^\rho \int_K \langle P^\pm \phi(ka), t^\pm f(k) \rangle dk . \tag{58}$$

We can rewrite the left hand side as follows:

$$\begin{aligned}
\langle c_{\gamma_\pm}(0)T^\pm \phi + (T^\pm)^w \hat{J}_0 \phi, t^\pm f \rangle &= \langle \pm c_\sigma(0)(T^\pm)^w \phi + (T^\pm)^w \hat{J}_0 \phi, t^\pm f \rangle \\
&= \langle \pm c_\sigma(0)\phi + \hat{J}_0 \phi, {}^t(T^\pm)^w t^\pm f \rangle \\
&= \langle \pm c_\sigma(0)\phi + \hat{J}_0 \phi, f \rangle .
\end{aligned}$$

We continue tranforming the right-hand side of (58).

$$\begin{aligned}
\langle \pm c_\sigma(0)\phi + \hat{J}_0 \phi, f \rangle &= \lim_{n\to\infty} \frac{1}{n} \int_0^n e^{t\rho} \int_K \langle P^\pm \phi(k \exp(tH)), t^\pm f(k) \rangle \, dk \, dt \\
&= \lim_{n\to\infty} \int_G \langle P\phi(x), f_n^\pm(x) \rangle \, dx,
\end{aligned}$$

where $H \in \mathfrak{a}_+$ is the unit vector, and the compactly supported sections $f_n^{\pm} \in L^2(X, V(\tilde{\gamma}_{\pm}, \tilde{\varphi}))$ are defined by $f_n^{\pm}(k_1 a k_2) := \frac{1}{n} a^{-\rho} \chi_{(0,n]}(\log(a)) \tilde{\gamma}_{\pm}(k_2^{-1}) t^{\pm} f(k_1)$. Here $\chi_{(0,n]}$ denotes the characteristic function of the interval $(0, n]$.

Assume now that $\text{supp}(f) \subset \Omega$ and consider $F_n^{\pm} \in L^2(Y, V_Y(\tilde{\gamma}_{\pm}, \tilde{\varphi}))$ given by

$$F_n^{\pm}(x) := \sum_{g \in \Gamma} \tilde{\varphi}(g) f_n^{\pm}(g^{-1} x) .$$

Then as in the proof of Proposition 9.2 we see that $\lim_{n \to \infty} F_n^{\pm} = 0$ in $L^2(Y, V_Y(\tilde{\gamma}_{\pm}, \tilde{\varphi}))$. Since $P^{\pm} \phi \in L^2(Y, V_Y(\gamma_{\pm}, \varphi))$ we obtain

$$\langle \pm c_{\sigma}(0)\phi + \hat{J}_0\phi, f \rangle = \lim_{n \to \infty} \int_{\Gamma \backslash G} \langle P^{\pm}\phi(x), F_n^{\pm}(x) \rangle \, dx = \lim_{n \to \infty} \langle P^{\pm}\phi, F_n^{\pm} \rangle = 0 .$$

Since $c_{\sigma}(0) \neq 0$ this proves $res(\phi) = 0$ and $res(\hat{J}_0\phi) = 0$. $\qquad\square$

The next lemma is independent of the theory of tempered invariant distribution vectors.

Lemma 9.6 *Let* $(\sigma, \lambda), (\tau, \mu) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*$, *and let*

$$A : C^{-\infty}(\partial X, V(\sigma_\lambda)) \to C^{-\infty}(\partial X, V(\tau_\mu))$$

be a G-intertwining operator. Let (φ, V_φ) be a finite-dimensional (not necessarily unitary) representation of Γ. We consider the operator

$$A_{\Gamma,\varphi} : {}^{\Gamma}C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to {}^{\Gamma}C^{-\infty}(\partial X, V(\tau_\mu, \varphi))$$

induced by $A \otimes \text{id}$. Then $\text{im}(A_{\Gamma,\varphi})$ is infinite-dimensional unless $\text{im}(A)$ is finite-dimensional.

Proof. The operator A restricts to a continuous operator

$$A : C^{\infty}(\partial X, V(\sigma_\lambda)) \to C^{\infty}(\partial X, V(\tau_\mu)) .$$

In fact, A induces an intertwining operator of the underlying $(\mathfrak{g}, K)$-modules which canonically extends to a continuous G-map between the spaces of smooth sections by the globalization theory of Casselman and Wallach [55], Ch. 11.

If f is a distribution section of a vector bundle over some manifold U, then let $\text{singsupp}(f) \subset U$ denote the singular support of f. We claim that $\text{singsupp} A(f) \subset \text{singsupp}(f)$ for all $f \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$.

Consider the delta distribution $\delta v \in C^{-\infty}(\partial X, V(\sigma_\lambda))$ at $x_0 := MAN \in \partial X$ with vector part $v \in V_\sigma$. Since δv is N-invariant we have $A(\delta v) \in {}^N C^{-\infty}(\partial X, V(\tau_\mu))$. The Bruhat decomposition $\partial X = Nwx_0 \cup \{x_0\}$ implies that $A(\delta v)$ is smooth outside x_0. Now let $f \in C^{-\infty}(\partial X, V(\sigma_\lambda))$.

Then there exists $\tilde{f} \in C_c^{-\infty}(G)$ such that $\mathrm{singsupp}(\tilde{f})P = \mathrm{singsupp}(f)$ and $\pi^{\sigma,\lambda}(\tilde{f})\delta v = f$. It follows that $A(f) = \pi^{\tau,\mu}(\tilde{f})A(\delta v)$, and hence $\mathrm{singsupp}\, A(f) \subset \mathrm{singsupp}(\tilde{f})P\{x_0\} = \mathrm{singsupp}(f)$. This shows the claim. In particular, $A_{\Gamma,\varphi}$ maps $^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ into $^\Gamma C_\Omega^{-\infty}(\partial X, V(\tau_\mu, \varphi))$ (see Lemma 6.6 for notation).

Assume now that $\dim \mathrm{im}(A_{\Gamma,\varphi}) < \infty$. By Lemma 6.6 the space $A_{\Gamma,\varphi}\left(^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))\right)$ is dense in $\mathrm{im}(A_{\Gamma,\varphi})$, hence $A_{\Gamma,\varphi}\left(^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))\right) = \mathrm{im}(A_{\Gamma,\varphi})$. We conclude that $\mathrm{im}(A_{\Gamma,\varphi}) \subset {}^\Gamma C_\Omega^{-\infty}(\partial X, V(\tau_\mu, \varphi))$.

Without loss of generality we may assume that $x_0 \in \Omega$. Choose $0 \neq w \in V_\varphi$ and consider the delta distribution

$$T := \sum_{g \in \Gamma} (\pi^{\sigma,\lambda}(g) \otimes \varphi(g))(\delta v \otimes w) \in {}^\Gamma C^{-\infty}(\Omega, V(\sigma_\lambda, \varphi)) \cong C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \ .$$

Since the singular parts of *ext* are finite-dimensional we find a smooth section $\phi \in C^\infty(B, V_B(\sigma_\lambda, \varphi))$ such that $f := ext(T - \phi) \in {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$ is defined. We decompose $\delta v \otimes w = f - (f - \delta v \otimes w)$. Since $x_0 \notin \mathrm{singsupp}(f - \delta v \otimes w)$ and $x_0 \notin \mathrm{singsupp}\, A_{\Gamma,\varphi}(f)$ we conlude that $A(\delta v)$ is smooth at x_0, and hence $A(\delta v) \in {}^N C^\infty(\partial X, V(\tau_\mu))$.

Now

$$^N C^\infty(\partial X, V(\tau_\mu)) = H^0(\mathfrak{n}, C^\infty(\partial X, V(\tau_\mu))) \cong H^0(\mathfrak{n}, C^\infty(\partial X, V(\tau_\mu))_K),$$

where the second equality is a special case of Casselman's comparison theorem for $\mathfrak{n}$-cohomology (see e.g. [12] or [27]). Thus $A(\delta v)$ is K-finite. There exists a finite-dimensional subspace $E \subset \mathcal{U}(\mathfrak{g})$ such that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{k})\mathcal{Z}E\mathcal{U}(\mathfrak{n})$ (see [54], 3.7.1). Thus the space $Z_K := \mathcal{U}(\mathfrak{g})A(\delta v) = \mathcal{U}(\mathfrak{k})EA(\delta v)$ is finite-dimensional.

Since δv generates $C^{-\infty}(\partial X, V(\sigma_\lambda))$, the element $A(\delta v)$ generates $\mathrm{im}(A)$. We conclude that $\mathrm{im}(A)_K = Z_K$ and thus $\dim \mathrm{im}(A) < \infty$. This finishes the proof of the lemma. $\square$

Corollary 9.7 *Let (φ, V_φ) be a finite-dimensional (not necessarily unitary) representation of Γ, and let V_π be an irreducible admissible representation of G. Then $^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)$ is infinite-dimensional unless V_π is finite-dimensional.*

Proof. By Casselman's subrepresentation theorem ([54], 3.8.3.) in conjunction with the functorial properties of the smooth globalization ([55], 11.6.7.) we find elements $(\sigma, \lambda), (\tau, \mu) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ such that $V_{\pi,-\infty}$ is a quotient of $C^{-\infty}(\partial X, V(\sigma_\lambda))$ and a submodule of $C^{-\infty}(\partial X, V(\tau_\mu))$. Thus there is a non-trivial G-intertwining operator

$$A : C^{-\infty}(\partial X, V(\sigma_\lambda)) \to C^{-\infty}(\partial X, V(\tau_\mu))$$

satisfying $\mathrm{im}(A) \cong V_{\pi,-\infty}$. If $\dim(V_\pi) = \infty$, then by Lemma 9.6 the subspace $\mathrm{im}(A_{\Gamma,\varphi}) \subset {}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)$ is infinite-dimensional. $\square$

The following proposition completes our description of the tempered and square-integrable invariant distribution vectors.

Proposition 9.8 *Let $(\pi, V_\pi) \in \hat{G}_d$. Then for any unitary representation (φ, V_φ) of Γ the space* $^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)_d$ *is infinite-dimensional.*

Proof. Let $\gamma \in \hat{K}$ be the minimal K-type of V_π. Casselman's subrepresentation theorem provides an embedding

$$\beta : V_{\pi,\infty} \to C^\infty(\partial X, V(\sigma_{-\lambda})) \ ,$$

where $\mathrm{Hom}_M(V_\sigma, V_\gamma) \neq 0$, and $-(\lambda + \rho)$ is the leading exponent in the asymptotic expansion for $a \to \infty$ of the matrix coefficients $c_{v,\tilde{v}}$, $v \in V_{\pi,\infty}$, $\tilde{v} \in V_{\pi',K}$. Since V_π is a discrete series representation we have $0 < \lambda \in \mathfrak{a}^*$. Forming the adjoint with respect to hermitian scalar products we obtain a projection

$$q : C^{-\infty}(\partial X, V(\sigma_\lambda)) \to V_{\pi,-\infty} \ .$$

By functoriality we can extend β to a map between the corresponding spaces of distribution vectors and obtain a G-intertwining operator

$$A := \beta \circ q : C^{-\infty}(\partial X, V(\sigma_\lambda)) \to C^{-\infty}(\partial X, V(\sigma_{-\lambda}))$$

satisfying $\mathrm{im}(A) \cong V_{\pi,-\infty}$. As in Lemma 9.6 we consider the operator

$$A_{\Gamma,\varphi} : {}^\Gamma C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to {}^\Gamma C^{-\infty}(\partial X, V(\sigma_{-\lambda}, \varphi)) \ .$$

This lemma combined with Lemma 6.6 tells us that $A_{\Gamma,\varphi}\left({}^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))\right)$ is infinite-dimensional. Hence $Z := (q \otimes \mathrm{id})\left({}^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))\right) \subset {}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)$ is infinite-dimensional, too. It remains to show that $Z \subset {}^\Gamma(V_{\pi,-\infty} \otimes V_\varphi)_d$.

Choose an embedding $t \in \mathrm{Hom}_K(V_{\tilde{\gamma}}, V_{\pi'})$, and define $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ by $\langle T(w), v \rangle := \langle w, [{}^t q \circ t(v)](e) \rangle$ for all $w \in V_\sigma$, $v \in V_{\tilde{\gamma}}$. Recall the definition of v_T from Lemma 9.1 and observe that $v_T = {}^t q(t(v))$. We consider the Poisson transform

$$P := P_\lambda^T \otimes \mathrm{id} : C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi)) \to C^\infty(X, V(\gamma, \varphi)) \ .$$

By Lemma 9.1 we find for all $\phi \in C^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$, $v \in V_{\tilde{\gamma}}$, $g \in G$

$$\langle P\phi(g), v \rangle = c_{\phi, v_T}(g) = c_{(q \otimes \mathrm{id})(\phi), t(v)}(g) \in V_\varphi \ . \tag{59}$$

Since $t(v)$ generates $V_{\pi',K}$ by Lemma 8.4 it suffices to show that $P\phi \in L^2(Y, V_Y(\gamma, \varphi))$ for all $\phi \in {}^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$.

Let D be a closed neighbourhood of $\mathrm{clo}(F) \cap \Omega$ for some fundamental domain $F \subset X$ of Γ, and let $Q \subset \partial X \setminus D$ be a closed neighbourhood of Λ. Let $\chi \in C^\infty(\partial X)$ be a cut-off function with $\mathrm{supp}(\chi) \subset Q$, $\mathrm{supp}(1 - \chi) \cap \Lambda = \emptyset$. Let $\phi \in {}^\Gamma C_\Omega^{-\infty}(\partial X, V(\sigma_\lambda, \varphi))$. By Lemma 6.2, 3., there exists a constant C such that for $a \gg 0$, $k \in DM$

$$|P(\chi\phi)(ka)| \leq Ca^{-(\lambda+\rho)} \ . \tag{60}$$

On the other hand $(1 - \chi)\phi \in C^\infty(\partial X, V(\sigma_\lambda, \varphi))$. By (59) the function $P((1 - \chi)\phi)$ can be expressed in terms of matrix coefficients $c_{s,t}$ of the discrete series representation V_π, where $s \in V_{\pi,\infty}$ and $t \in V_{\pi',K}$. Thus $P((1-\chi)\phi)$ has an asymptotic expansion with leading exponent $-(\lambda+\rho)$. In particular, it satisfies an estimate of the form (60), too. We conclude that $P\phi \in L^2(Y, V_Y(\gamma, \varphi))$. This finishes the proof of the proposition. $\qquad\qquad\square$

10   Eisenstein series, wave packets, and scalar products

In this section we consider the Eisenstein series and the wave-packet transform. For a moment we can drop the unitarity condition on φ. Let γ be a finite-dimensional unitary representation of K, $\sigma \in \hat{M}$, $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$, and let P_λ^T, $\lambda \in \mathfrak{a}_\mathbb{C}^*$, be the associated Poisson transform (see Definition 4.8).

Definition 10.1 *For* $\phi \in C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ *we define the Eisenstein series* $E(\lambda, \phi, T) \in C^\infty(Y, V_Y(\gamma, \varphi))$ *by*
$$E(\lambda, \phi, T) := P_\lambda^T \circ ext(\phi) \ .$$

The Eisenstein series $E(\lambda, \phi, T)$ is an eigenvector of $\mathcal{Z}$ for the infinitesimal character of the principal series representation $\pi^{\sigma,\lambda}$. Theorem 5.10 and the functional equation of the Poisson transform (18) have the following immediate corollary.

Corollary 10.2 *The Eisenstein series gives rise to a meromorphic family defined on* $\mathfrak{a}_\mathbb{C}^*$ *(or* $\{\mathrm{Re}(\lambda) > \delta_\Gamma + \delta_\varphi\}$ *in case that* $X = \mathbb{O}H^2$*) of continuous maps*
$$E(\lambda, ., T) : C^{-\infty}(B, V_B(\sigma_\lambda, \varphi)) \to C^\infty(Y, V_Y(\gamma, \varphi))$$
with finite-dimensional singularities. It satisfies the functional equation
$$E(\lambda, \hat{S}_{-\lambda}\phi, T) = E(-\lambda, \phi, (c_\gamma(\lambda)T)^w) \ . \tag{61}$$

From Lemma 6.2, 3., we gain detailed knowledge of the asymptotics of $E(\lambda, \phi, T)(y)$ for $y \to b \in B$ (see the discussion of (66) in the proof of Proposition 10.8 below).

We now return to our unitarity assumption on φ.

Corollary 10.3

1. *The Eisenstein series is regular on* $\{\mathrm{Re}(\lambda) = 0, \lambda \neq 0\}$.

2. *If* $\mathrm{Re}(\lambda) = 0$, $\lambda \neq 0$, *then* $E(\lambda, ., T)$ *maps* $C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ *continuously to* $\mathcal{C}'(Y, V_Y(\gamma, \varphi))$.

3. *In the half plane $\{\mathrm{Re}(\mu) \geq 0\}$ the Eisenstein series has at most first-order poles which are located in the finite set $PS_{\mathrm{res}}(\sigma, \varphi) \subset [0, \delta_\Gamma]$. The residue at $\lambda \in PS_{\mathrm{res}}(\sigma, \varphi) \setminus \{0\}$*

$$\mathrm{res}_{\mu=\lambda} E(\mu, ., T)$$

maps $C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ to $L^2(Y, V_Y(\gamma, \varphi))$.

Proof. 1. follows from Lemma 7.2. To see 2. note that $\pi^{\sigma, \lambda}$ is tempered if $\mathrm{Re}(\lambda) = 0$, that $E(\lambda, \phi, T)$ can be expressed in terms of matrix coefficients of $\pi^{\sigma, \lambda}$ (Lemma 9.1), and apply 9.4. We have

$$\mathrm{res}_{\mu=\lambda} E(\mu, ., T)(\phi) = P_\lambda^T \circ (\mathrm{res}_\lambda ext)(\phi) \ . \tag{62}$$

3. now follows from Propositions 7.3, 7.4, and 7.8. $\qquad\square$

By (62) the following proposition can also be considered as the determination of the L^2-scalar product between residues of Eisenstein series.

We introduce a scalar product on $\mathrm{Hom}_M(V_\sigma, V_\gamma)$ by $(T_1, T_2)\mathrm{id}_{V_\sigma} := T_2^* T_1$. This makes sense because of our standing assumption $\sigma \in \hat{M}$. By $(., .)_B$ we denote the natural sesquilinear pairing between $C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ and $C^\infty(B, V_B(\sigma_{-\bar\lambda}, \varphi))$.

Proposition 10.4 *Let $\lambda \in PS_{\mathrm{res}}(\sigma, \varphi) \setminus \{0\}$, let γ be a finite-dimensional representation of K, $T_1, T_2 \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$, $\phi_1 = (\mathrm{res}_\lambda ext)(f) \in E_\Lambda(\sigma_\lambda, \varphi)$ for some $f \in C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$, and $\phi_2 \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_\lambda, \varphi))$. Then*

$$(P_\lambda^{T_1} \phi_1, P_\lambda^{T_2} \phi_2)_{L^2(Y, V_Y(\gamma, \varphi))} = \omega_X \left(c_\gamma(\lambda) T_1, T_2^w \right) (f, res \circ \hat{J}_\lambda \phi_2)_B \ .$$

Here $\omega_X = \frac{\omega_n}{2^r \alpha^{n-1}}$, where $n = \dim X$, $\omega_n = \mathrm{vol}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ and $r \in \mathbb{N}$ is such that $r\alpha = 2\rho$. If γ is the minimal K-type of $\pi^{\sigma, \lambda}$, then

$$(c_\gamma(\lambda) T_1, T_2^w) = c_\sigma(\lambda)(T_1, T_2) \ .$$

Proof. First we can assume that f is smooth. We extend f to a holomorphic family $\mu \mapsto f_\mu \in C^\infty(B, V_B(\sigma_\mu, \varphi))$ defined in a neighbourhood of λ. Let $B_R \subset X$ be the ball of radius R, and let $(., .)_{B_R}$ denote the scalar product in $L^2(B_R, V(\gamma, \varphi))$. Let $\chi \in C^\infty(X)$ be the cut-off function constructed in Lemma 6.4. Set

$$S_R(\mu) := (\mu - \lambda) \left(P_\mu^{T_1} ext(f_\mu), \chi P_\lambda^{T_2} \phi_2 \right)_{B_R} \ .$$

Then

$$(P_\lambda^{T_1} \phi_1, P_\lambda^{T_2} \phi_2)_{L^2(Y, V_Y(\gamma, \varphi))} = \lim_{R \to \infty} \lim_{\mu \to \lambda} S_R(\mu) \ .$$

Let $D := -\Omega_G + c(\sigma) + \lambda^2$ be the shifted Casimir operator. As in the proof of Proposition 7.3 we obtain

$$S_R(\mu) = -\frac{1}{\lambda + \mu} \left(D P_\mu^{T_1} ext(f_\mu), \chi P_\lambda^{T_2} \phi_2 \right)_{B_R} \ .$$

While $P_\mu^{T_1} ext(f_\mu)$ has a first order pole at $\mu = \lambda$, $DP_\mu^{T_1} ext(f_\mu)$ is regular, and its value at $\mu = \lambda$ equals DF, where $F \in {}^\Gamma C^\infty(X, V(\gamma, \varphi))$ is the constant term of the Laurent expansion of $P_\mu^{T_1} ext(f_\mu)$ at λ.

Note that $D = \nabla^* \nabla + \mathcal{R}$ for some selfadjoint endomorphism $\mathcal{R}$ of $V(\gamma, \varphi)$, where $\nabla^* \nabla$ is the Bochner Laplacian associated to the invariant connection ∇ of $V(\gamma, \varphi)$. Thus we can apply Green's formula in the same spirit as in the proof of Proposition 6.5:

$$
\begin{aligned}
\lim_{\mu \to \lambda} S_R(\mu) &= -\frac{1}{2\lambda} \left((DF, \chi P_\lambda^{T_2} \phi_2)_{B_R} - (F, D\chi P_\lambda^{T_2} \phi_2)_{B_R} + (F, [D, \chi] P_\lambda^{T_2} \phi_2)_{B_R} \right) \\
&= \frac{1}{2\lambda} \left((\nabla_n F, \chi P_\lambda^{T_2} \phi_2)_{\partial B_R} - (F, \nabla_n \chi P_\lambda^{T_2} \phi_2)_{\partial B_R} - (F, [D, \chi] P_\lambda^{T_2} \phi_2)_{B_R} \right) . \quad (63)
\end{aligned}
$$

Applying Cauchy's integral formula to the asymptotic expansion of $P_\mu^{T_1} ext(f_\mu)$ given in Lemma 6.2, 3. we find some $\epsilon > 0$ such that for $a \to \infty$

$$
F(ka) = a^{\lambda - \rho} c_\gamma(\lambda) T_1 f(k) + O(a^{\lambda - \rho - \epsilon})
$$

uniformly as kM varies in compact subsets of Ω.

As in the proof of Proposition 6.5 we obtain $(F, [D, \chi] P_\lambda^{T_2} \phi_2)_X = 0$. In order to perform the limit $R \to \infty$ in (63) we use $\lim_{R \to \infty} e^{-2\rho R} \mathrm{vol}(\partial B_R) = \omega_X$ and the asymptotic expansions of F, $P_\lambda^{T_2} \phi_2$, and obtain

$$
\begin{aligned}
(P_\lambda^{T_1} \phi_1, P_\lambda^{T_2} \phi_2)_{L^2(Y, V_Y(\gamma, \varphi))} &= \frac{\omega_X}{2\lambda} \left((\lambda - \rho) \int_{\partial X} (c_\gamma(\lambda) T_1 f(k), \chi_\infty(k) T_2^w (\hat{J}_\lambda \phi_2)(k)) \, dk \right. \\
&\qquad \left. + (\lambda + \rho) \int_{\partial X} (c_\gamma(\lambda) T_1 f(k), \chi_\infty(k) T_2^w (\hat{J}_\lambda \phi_2)(k)) \, dk \right) \\
&= \omega_X (T_2^{w*} c_\gamma(\lambda) T_1 f, res \circ \hat{J}_\lambda \phi_2)_B .
\end{aligned}
$$

This finishes the proof of the proposition. $\square$

On $I^{\sigma, \lambda}$ we consider consider the scalar product (56). Let $(.,.)$ be the scalar product on $E_\Lambda(\sigma_\lambda, \varphi) \oplus U_\Lambda(\sigma_\lambda, \varphi) = {}^\Gamma (I_{-\infty}^{\sigma, \lambda} \otimes V_\varphi)_d$ induced by the matrix coefficient map (see Proposition 9.2 and Corollary 8.7).

Corollary 10.5 *The decomposition $E_\Lambda(\sigma_\lambda, \varphi) \oplus U_\Lambda(\sigma_\lambda, \varphi)$ is orthogonal with respect to $(.,.)$. If $\phi_1, \phi_2 \in E_\Lambda(\sigma_\lambda, \varphi)$, then*

$$
(\phi_1, \phi_2) = \frac{\omega_X c_\sigma(\lambda)}{\dim(V_\sigma)} (f, res \circ \hat{J}_\lambda \phi_2) ,
$$

where $f \in C^{-\infty}(B, V_B(\sigma_\lambda, \varphi))$ such that $\phi_1 = (res_\lambda ext)(f)$.

Proof. Let γ be the minimal K-type of $\pi^{\sigma, \lambda}$. For $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ and $v \in V_{\tilde\gamma}$ let $v_T \in C^\infty(\partial X, V(\tilde\sigma_{\pm\lambda}))$ be given by $v_T(k) = {}^t T \tilde\gamma(k^{-1}) v$. We have $J_{\tilde\sigma, -\lambda} v_T = v_T$. In fact, σ is Weyl-invariant and thus for all $\phi \in C^{-\infty}(\partial X, V(\sigma_{-\lambda}))$

$$
\langle \phi, J_{\tilde\sigma, -\lambda} v_T \rangle = \langle J_{\sigma, -\lambda} \phi, v_T \rangle
$$

$$
\begin{aligned}
&= & \langle \hat{J}_{\sigma,-\lambda} c_\sigma(\lambda)^{-1}\phi, v_T \rangle \\
&\overset{9.1}{=} & c_\sigma(\lambda)^{-1}\langle P_\lambda^T \circ \hat{J}_{\sigma,-\lambda}(\phi)(1), v \rangle \\
&\overset{(18)}{=} & c_\sigma(\lambda)^{-1}\langle P_{-\lambda}^{(c_\gamma(\lambda)T)^w}(\phi)(1), v \rangle \\
&\overset{(22)}{=} & \langle P_{-\lambda}^T(\phi)(1), v \rangle \\
&= & \langle \phi, v_T \rangle \ .
\end{aligned}
$$

By (56) we have

$$
\begin{aligned}
([v_T],[v_T]) &=& (v_T, J_{\tilde{\sigma},-\lambda} v_T)_0 \\
&=& \int_K \|v_T(k)\|^2 dk \\
&=& \int_K ({}^t T\tilde{\gamma}(k^{-1})v, {}^t T\tilde{\gamma}(k^{-1})v)dk \\
&=& \int_K (\tilde{\gamma}(k)({}^t T)^{*t}T\tilde{\gamma}(k^{-1})v, v)dk \\
&=& \frac{\dim(V_{\tilde{\sigma}})}{\dim(V_{\tilde{\gamma}})}\|v\|^2\|T\|^2 \ .
\end{aligned}
\tag{64}
$$

Let $\{v^i\}_{i=1}^{\dim(V_{\tilde{\gamma}})}$ be an othonormal base of $V_{\tilde{\gamma}}$. We compute using Proposition 10.4, Lemma 9.1

$$
\begin{aligned}
\frac{\omega_X c_\sigma(\lambda)}{\dim(V_\sigma)}(f, res \circ \hat{J}_\lambda \phi_2) &=& \frac{1}{\|T\|^2 \dim(V_\sigma)}(P_\lambda^T \phi_1, P_\lambda^T \phi_2)_{L^2(Y, V_Y(\gamma,\varphi))} \\
&=& \frac{1}{\|T\|^2 \dim(V_\sigma)}\sum_{i=1}^{\dim(V_\gamma)}(c_{\phi_1, v_T^i}, c_{\phi_2, v_T^i})_{L^2(\Gamma\backslash G,\varphi)} \\
&=& \frac{1}{\|T\|^2 \dim(V_\sigma)}\sum_{i=1}^{\dim(V_\gamma)}(\phi_1, \phi_2)\|v_T^i\|^2 \\
&=& (\phi_1, \phi_2) \ .
\end{aligned}
$$

This proves the corollary. $\qquad\square$

Now we turn to the definition of the wave packet transform. Roughly speaking, a wave packet of Eisenstein series is an average of the Eisenstein series over imaginary parameters with, say, a smooth, compactly supported weight function with respect to the expected Plancherel measure $p_\sigma(i\mu)d\mu$. More precisely, let $\mathfrak{a}_+^* := \{\lambda \in \mathfrak{a}^* \mid \langle \lambda, \alpha \rangle > 0\}$ be the open positive chamber in $\mathfrak{a}^*$. Then the space of such weight functions $\mathcal{H}_0^\sigma(\varphi)$ is the linear space of smooth families $\mathfrak{a}_+^* \ni \mu \mapsto \phi_{i\mu} \in C^\infty(B, V_B(\sigma_{i\mu}, \varphi))$ with compact support in $\mathfrak{a}_+^*$ with respect to μ. Because of the functional equation (61) it will be sufficient to consider wave packets on the positive imaginary axis, only.

Let γ be a finite-dimensional unitary representation of K. We first define the wave packet transform on $\mathcal{H}_0^\sigma(\varphi) \otimes \text{Hom}_M(V_\sigma, V_\gamma)$. Later we will extend it by continuity to a Hilbert space closure.

Definition 10.6 *The wave packet transform is the map*

$$E : \mathcal{H}_0^\sigma(\varphi) \otimes \mathrm{Hom}_M(V_\sigma, V_\gamma) \to C^\infty(Y, V_Y(\gamma, \varphi))$$

given by

$$E(\phi \otimes T) := E(\phi, T) := \int_{\mathfrak{a}_+^*} E(i\mu, \phi_{i\mu}, T) p_\sigma(i\mu) \, d\mu \ ,$$

where $d\mu$ is the Lebesgue measure on $\mathfrak{a}_+^* \cong (0, \infty)$. The section $E(\phi, T)$, $\phi \in \mathcal{H}_0^\sigma(\varphi)$, $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ is called a wave packet (of Eisenstein series).

Lemma 10.7 *If $T \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$, $\phi \in \mathcal{H}_0^\sigma(\varphi)$, then $E(\phi, T) \in \mathcal{C}(Y, V_Y(\gamma, \varphi))$.*

Proof.   Set $\psi_{i\mu} := ext \, \phi_{i\mu}$ and define

$$P(\psi) := \int_{\mathfrak{a}_+^*} P_{i\mu}^T(\psi_{i\mu}) p_\sigma(i\mu) \, d\mu \ .$$

Let χ be the cut-off function constructed in Lemma 6.4. In view of Definition 8.1 of the Schwartz space we have to show that $\chi P(\psi) \in \mathcal{C}(X, V(\gamma, \varphi))$. Let $\chi_0 \in C_c^\infty(X)$ be some cut-off function which is equal to 1 on some neighbourhood of $eK \in X$. Obviously, we have $\chi \chi_0 P(\psi) \in C_c^\infty(X, V(\gamma, \varphi)) \subset \mathcal{C}(X, V(\gamma, \varphi))$. It remains to show that

$$\chi_1 P(\psi) \in \mathcal{C}(X, V(\gamma, \varphi)) \ ,$$

where $\chi_1 = \chi(1 - \chi_0)$. Observe that the seminorms $q_{D,N}$, $D \in \mathcal{U}(\mathfrak{g})$, $N \in N_0$ defined by

$$q_{D,N}(f)^2 := \int_G |\log(a_g)^N f(Dg)|^2 \, dg$$

are sufficient in order to define the topology on $\mathcal{C}(X, V(\gamma, \varphi))$. We thus have to show that $g \mapsto \log(a_g)^N(\chi_1 P(\psi))(Dg)$ is square-integrable. Let $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ denote the coproduct and write $\Delta(D) = \sum_\alpha D_\alpha \otimes D_\alpha'$ for some $D_\alpha, D_\alpha' \in \mathcal{U}(\mathfrak{g})$. Then $(\chi_1 P(\psi))(Dg) = \sum_\alpha \chi_1(D_\alpha g) P(\psi)(D_\alpha' g)$. By Lemma 6.4, 4., $g \mapsto \chi_1(Dg)$ is bounded for any $D \in \mathcal{U}(\mathfrak{g})$. It remains to show for any $D \in \mathcal{U}(\mathfrak{g})$ and cut-off function $\chi_2 \in C^\infty(X)$ having compact support in $(X \cup \Omega) \setminus \{eK\}$ the function

$$\log(a_g)^N \chi_2(g) P(\psi)(Dg) = \log(a_g)^N \chi_2(ka) \int_{\mathfrak{a}_+^*} P_{i\mu}^T(\pi^{\sigma, i\mu}(D)\psi_{i\mu})(g) p_\sigma(i\mu) \, d\mu \qquad (65)$$

is square integrable. By Lemma 6.2, 3., there exists $\epsilon > 0$ such that for $D \in \mathcal{U}(\mathfrak{g})$, $ka \in \mathrm{supp}(\chi_2)$

$$
\begin{aligned}
P_{i\mu}^T(\pi^{\sigma, i\mu}(D)\psi_{i\mu})(ka) \ = \ & a^{i\mu - \rho} c_\gamma(i\mu) T \psi_{i\mu}(k) \\
& + a^{-i\mu - \rho} T^w(\hat{J}_{i\mu}(\pi^{\sigma, i\mu}(D)\psi_{i\mu}))(k) \\
& + a^{-(\rho + \epsilon)} R(i\mu, \pi^{\sigma, i\mu}(D)\psi_{i\mu}, ka) \ ,
\end{aligned}
\qquad (66)
$$

where the remainder function $(\mu, ka) \mapsto R(i\mu, \pi^{\sigma, i\mu}(D)\psi_{i\mu}, ka)$ is uniformly bounded.

Since the families $\psi_{i\mu}$ and $\hat{J}_{i\mu}(\pi^{\sigma, i\mu}(D)\psi_{i\mu})$ have compact support with respect to μ and are smooth in (μ, k) (as long $kM \in \Omega$) by Lemma 5.3 each summand of (66) contributes to the function (65) a summand which is bounded by $C_{N'}(1 + \log(a_g))^{-N'} a_g^{-\rho}$ for any $N' \in \mathbb{N}$. It follows that the function (65) is square integrable. This implies the lemma. $\qquad \square$

Let $\sigma^i \in \hat{M}$, $T_i \in \mathrm{Hom}_M(V_{\sigma^i}, V_\gamma)$, $i = 1, 2$, $\lambda \in \mathfrak{a}_+^*$, $\phi \in \mathcal{H}_0^{\sigma^2}(\varphi)$, and $\psi \in C^{-\infty}(B, V_B(\sigma^1{}_{i\lambda}, \varphi))$. By Lemma 10.7 and Corollary 10.3 the pairing $(E(i\lambda, \psi, T_1), E(\phi, T_2))_{L^2(Y, V_Y(\gamma, \varphi))}$ between the Eisenstein series $E(i\lambda, \psi, T_1)$ and the wave packet $E(\phi, T_2)$ is well-defined. The following proposition gives an explicit formula for this pairing and is the crucial step in the determination of the absolute continuous part of the Plancherel measure.

Proposition 10.8 *We have*

$$(E(i\lambda, \psi, T_1), E(\phi, T_2))_{L^2(Y, V_Y(\gamma, \varphi))} = \begin{cases} 2\pi\omega_X (T_1, T_2) (\psi, \phi_{i\lambda})_B & \sigma^1 = \sigma^2 \\ 0 & \sigma^1 \neq \sigma^2 \end{cases}.$$

Proof. For fixed ψ and T_i, $i = 1, 2$, we consider the continuous linear functional (which is in fact a distribution section of a bundle over $i\mathfrak{a}_+^* \times B$)

$$R : \mathcal{H}_0^{\sigma^2}(\varphi) \to \mathbb{C}$$

given by $R(\phi) := (E(i\lambda, \psi, T_1), E(\phi, T_2))_{L^2(Y, V_Y(\gamma, \varphi))}$. If $D \in \mathcal{Z}$ and $\sigma \in \hat{M}$ we consider the polynomial $\chi_{\mu_\sigma + \rho_m - \mu}(D)$ on $\mathfrak{a}_\mathbb{C}^*$. Since $\chi_{\mu_\sigma + \rho_m - \mu}$ is the infinitesimal character of $\pi^{\sigma, \mu}$ we have

$$\begin{aligned} 0 &= ((D^* - \chi_{\mu_{\sigma^1} + \rho_m - i\lambda}(D^*))E(i\lambda, \psi, T_1), E(\phi, T_2))_{L^2(Y, V_Y(\gamma, \varphi))} \\ &= (E(i\lambda, \psi, T_1), (D - \bar{\chi}_{\mu_{\sigma^1} + \rho_m - i\lambda}(D^*))E(\phi, T_2))_{L^2(Y, V_Y(\gamma, \varphi))} \\ &= (E(i\lambda, \psi, T_1), E(\tilde{\phi}, T_2))_{L^2(Y, V_Y(\gamma, \varphi))} \,, \end{aligned}$$

where $\tilde{\phi}_{i\mu} = (\chi_{\mu_{\sigma^2} + \rho_m - i\mu}(D) - \chi_{\mu_{\sigma^1} + \rho_m - i\lambda}(D))\phi_{i\mu}$. We conclude that multiplication by the polynomial $(\chi_{\mu_{\sigma^2} + \rho_m - i\mu}(D) - \chi_{\mu_{\sigma^1} + \rho_m - i\lambda}(D))$ annihilates the functional R. Thus R is supported on the zero set of this polynomial. If $\sigma^1 \neq \sigma^2$, then

$$\bigcap_{D \in \mathcal{Z}} \{\mu \in \mathfrak{a}_+^* \mid (\chi_{\mu_{\sigma^2} + \rho_m - i\mu}(D) - \chi_{\mu_{\sigma^1} + \rho_m - i\lambda}(D)) = 0\} = \emptyset$$

and therefore $R = 0$. This proves the proposition in case $\sigma^1 \neq \sigma^2$.

Assume now that $\sigma^1 = \sigma^2 =: \sigma$. Observe that

$$\bigcap_{D \in \mathcal{Z}} \{\mu \in \mathfrak{a}_+^* \mid (\chi_{\mu_\sigma + \rho_m - i\mu}(D) - \chi_{\mu_\sigma + \rho_m - i\lambda}(D)) = 0\} = \{\lambda\}$$

and that the functional R is of the form $R(\phi) = r(\phi_{i\lambda})$ for some $r \in C^{-\infty}(B, V_B(\sigma_{i\lambda}, \varphi))$, where r remains to be determined. We prefer to give a direct proof of the proposition in case $\sigma^1 = \sigma^2$, which does not refer to this observation.

Because of the continuity of $E(i\lambda, ., T_1)$ (see Corollary 10.3, 2.) we can assume that $\psi \in C^\infty(B, V_B(\sigma_{i\lambda}, \varphi))$. We apply Green's formula in a similar way as in the proofs of Propositions 6.5 and 10.4.

Let χ be the cut-off function as constructed in Lemma 6.4 and B_R the ball of radius R around the origin of X. For $\mu \in \mathfrak{a}_+^*$ we consider

$$S_R(\mu) := (E(i\lambda, \psi, T_1), \chi E(i\mu, \phi_{i\mu}, T_1))_{B_R} .$$

We have thus to compute

$$\lim_{R \to \infty} \int_0^\infty S_R(\mu) p_\sigma(i\mu) \, d\mu .$$

If we set $A := -\Omega_G + c(\sigma)$, then we obtain

$$
\begin{aligned}
(\lambda^2 - \mu^2) S_R(\mu) &= (A\, E(i\lambda, \psi, T_1), \chi E(i\mu, \phi_{i\mu}, T_2))_{B_R} - (E(i\lambda, \psi, T_1), A\chi E(i\mu, \phi_{i\mu}, T_2))_{B_R} \\
&\quad + (E(i\lambda, \psi, T_1), [A, \chi] E(i\mu, \phi_{i\mu}, T_2))_{B_R} \\
&= -(\nabla_n E(i\lambda, \psi, T_1), \chi E(i\mu, \phi_{i\mu}, T_2))_{\partial B_R} \quad\quad\quad\quad\quad\quad (67) \\
&\quad + (E(i\lambda, \psi, T_1), \nabla_n \chi E(i\mu, \phi_{i\mu}, T_2))_{\partial B_R} \quad\quad\quad\quad\quad (68) \\
&\quad + ([A, \chi] E(i\mu, \phi_{i\mu}, T_1), E(i\lambda, \psi, T_2))_{B_R} . \quad\quad\quad\quad\quad (69)
\end{aligned}
$$

We now apply the asymptotic expansion (66) which holds on the support of χ in case that $D = 1$, and for $ext(\phi_{i\mu})$ and $ext(\psi)$ in place of $\psi_{i\mu}$. Note that for $k \in \Omega M$ we have $ext(\phi_{i\mu})(k) = \phi_{i\mu}(k)$, $\hat{J}_{i\mu}(ext(\phi_{i\mu}))(k) = \hat{S}_{i\mu}(\phi_{i\mu})(k)$, etc. We obtain for large R with $a_R := \mathrm{e}^R$, $\omega_R := a_R^{-2\rho} \mathrm{vol}(\partial B_R)$

$$
\begin{aligned}
\frac{1}{\omega_R}((67) + (68)) &= i(-\lambda - \mu) a_R^{i(\lambda - \mu)} \int_K \chi_\infty(k)(c_\gamma(i\lambda) T_1 \psi)(k), c_\gamma(i\mu) T_2 \phi_{i\mu}(k)) \, dk \\
&\quad + i(-\lambda + \mu) a_R^{i(\lambda + \mu)} \int_K \chi_\infty(k)(c_\gamma(i\lambda) T_1 \psi(k), T_2^w(\hat{S}_{i\mu}\phi_{i\mu})(k)) \, dk \\
&\quad + i(\lambda - \mu) a_R^{i(-\lambda - \mu)} \int_K \chi_\infty(k)(T_1^w \hat{S}_{i\lambda}\psi)(k), c_\gamma(i\mu) T_2 \phi_{i\mu}(k)) \, dk \\
&\quad + i(\lambda + \mu) a_R^{i(-\lambda + \mu)} \int_K \chi_\infty(k)(T_1^w(\hat{S}_{i\lambda}\psi)(k), T_2^w(\hat{S}_{i\mu}\phi_{i\mu})(k)) \, dk \\
&\quad + o(1) .
\end{aligned}
$$

The remainder term $o(1)$ contains integrals over K of terms involving the normal derivative of χ, the difference $\chi(k a_R) - \chi_\infty(k)$ and the function $a_R^{-\rho - \epsilon} R(i\mu, ., k a_R)$ appearing in (66). We combine this remainder with the term (69) divided by ω_R to $F(\lambda, \mu, R)$. Since the asymptotic expansion (66) can be differentiated with respect to μ, there exists a constant $C \in \mathbb{R}$ such that

$$|F(\lambda, \mu, R)| + |\frac{d}{d\mu} F(\lambda, \mu, R)| < C, \quad \forall R > 0, \, \mu \in \mathfrak{a}_+^* . \quad\quad\quad (70)$$

We can write for $\mu \neq \lambda$

$$
\begin{aligned}
\frac{1}{\omega_R} S_R(\mu) &= i\frac{a_R^{i(\lambda-\mu)}}{-\lambda+\mu}(T_2^* c_\gamma(i\mu)^* c_\gamma(i\lambda)T_1\psi, \phi_{i\mu})_B + i\frac{a_R^{i(\lambda+\mu)}}{-\lambda-\mu}(T_2^{w*} c_\gamma(i\lambda)T_1\psi, \hat{S}_{i\mu}\phi_{i\mu})_B \\
&\quad + i\frac{a_R^{i(-\lambda-\mu)}}{\lambda+\mu}((\hat{S}_{i\lambda}\psi, T_1^{w*}c_\gamma(i\mu)T_2\phi_{i\mu}))_B + i\frac{a_R^{i(-\lambda+\mu)}}{\lambda-\mu}(T_2^{w*}T_1^{w}\hat{S}_{i\lambda}\psi, \hat{S}_{i\mu}\phi_{i\mu})_B \quad (71) \\
&\quad + \frac{F(\lambda,\mu,R)}{\lambda^2-\mu^2} \ .
\end{aligned}
$$

By the Lemma of Riemann-Lebesgue

$$
\lim_{R\to\infty}\int_0^\infty \frac{a_R^{i(\lambda+\mu)}}{-\lambda-\mu}(T_2^{w*}c_\gamma(i\lambda)T_1\psi, \hat{S}_{i\mu}\phi_{i\mu})_B\, p_\sigma(i\mu)\, d\mu = 0 \ ,
$$

$$
\lim_{R\to\infty}\int_0^\infty \frac{a_R^{i(-\lambda-\mu)}}{\lambda+\mu}(\hat{S}_{i\lambda}\psi, T_1^{w*}c_\gamma(i\mu)T_2\phi_{i\mu})_B\, p_\sigma(i\mu)\, d\mu = 0 \ .
$$

We set $s := \mu - \lambda$ and regroup the remaining terms of (71) to

$$
\frac{a_R^{is} - a_R^{-is}}{is}(T_2^* c_\gamma(i\mu)^* c_\gamma(i\lambda)T_1\psi, \phi_{i\mu})_B \tag{72}
$$

$$
- \ a_R^{is}\frac{1}{is}\left((T_2^* c_\gamma(i\mu)^* c_\gamma(i\lambda)T_1\psi, \phi_{i\mu})_B - (T_1,T_2)(\hat{S}_{i\lambda}\psi, \hat{S}_{i\mu}\phi_{i\mu})_B\right) \tag{73}
$$

$$
+ \ \frac{F(\lambda,\mu,R)}{\lambda^2-\mu^2} \ .
$$

Note that (72) is smooth at $\mu = \lambda$. We claim that (73) is smooth at $\mu = \lambda$, too.

By $\hat{S}_{i\lambda}^* = \hat{S}_{-i\lambda}$ and the functional equation of the scattering matrix (30) we obtain

$$
\hat{S}_{i\lambda}^* \hat{S}_{i\lambda} = \frac{1}{p_\sigma(i\lambda)}\mathrm{id} \ .
$$

The claim now follows from

$$
T_2^* c_\gamma(i\lambda)^* c_\gamma(i\lambda)T_1 = \frac{(T_1,T_2)}{p_\sigma(i\lambda)} \tag{74}
$$

which is a consequence of (20) and (21).

Now (71) forces also $\frac{F(\lambda,\mu,R)}{\lambda^2-\mu^2}$ to be smooth at $\lambda = \mu$. By (70) and Lebesgue's theorem about dominated convergence we obtain

$$
\lim_{R\to\infty}\int_0^\infty \frac{F(\lambda,\mu,R)}{\lambda^2-\mu^2}\, p_\sigma(i\mu)\, d\mu = 0 \ .
$$

If we integrate (73) with respect to s and perform the limit $R \to \infty$, then the result vanishes by the Riemann-Lebesgue lemma.

We now use (74) and the identity of distributions $\lim_{r\to\infty}\frac{\sin(rs)}{s} = \pi\delta_0(s)$ in order to compute

$$
\begin{aligned}
\lim_{R\to\infty}\int_0^\infty S_R(\mu)p_\sigma(i\mu)\, d\mu &= \lim_{R\to\infty}\omega_R\int_{-\infty}^\infty \frac{a_R^{is} - a_R^{-is}}{is}(T_2^* c_\gamma(i\mu)^* c_\gamma(i\lambda)T_1\psi, \phi_{i\mu})_B\, p_\sigma(i\mu)\, ds \\
&= 2\pi\,\omega_X\,(T_1,T_2)\,(\psi, \phi_{i\lambda})_B \ .
\end{aligned}
$$

This proves the proposition.   □

11   The Plancherel theorem and spectral decompositions

In this final section we obtain our explicit Plancherel theorem, i.e., the decomposition of $L^2(\Gamma\backslash G, \varphi)$. We use the scalar product formula of Proposition 10.8 in order to show that the subspace of $L^2(\Gamma\backslash G, \varphi)$ spanned by the wave packets of Eisenstein series is the absolute continuous subspace, that its complement is the discrete subspace, and that there is no singular continuous subspace. It is not surprising that the absolute continuous part of the Plancherel measure κ coincides with the absolute continuous part for $L^2(G)$.

As a consequence of the decomposition of the Plancherel theorem we derive the spectral decomposition of $L^2(Y, V_Y(\gamma, \varphi))$ with respect to the invariant differential operators.

We first introduce and describe certain subspaces of $\int_{\hat{G}}^{\oplus} M_\pi \hat{\otimes} V_\pi \, d\kappa(\pi)$ corresponding to the partition $\hat{G} = \hat{G}_d \cup \hat{G}_u \cup \hat{G}_c$ (see the beginning of Section 9).

For each $\pi \in \hat{G}$ we fix the scalar product on ${}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d$ such that the matrix coefficient map c_π (see Corollary 8.7) is unitary.

We define the Hilbert space associated to discrete series $\hat{G}_d$ by

$$\mathcal{H}_{cusp}(\varphi) := \overset{\text{Hilbert}}{\underset{\pi \in \hat{G}_d}{\bigoplus}} {}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d \hat{\otimes} V_\pi \ .$$

According to Proposition 9.2 and Corollary 10.5 for $\pi = \bar{\pi}^{\sigma,\lambda} \in \hat{G}_c$ we have an orthogonal decomposition

$${}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d = C^{-\infty}(\Lambda, V(\tilde{\sigma}_\lambda)) = E_\Lambda(\tilde{\sigma}_\lambda, \varphi) \oplus U_\Lambda(\tilde{\sigma}_\lambda, \varphi) \ .$$

The same corollary gives an alternative expression for the restriction of the scalar product to $E_\Lambda(\sigma_\lambda, \varphi)$ in terms of the boundary geometry. This space is non-tivial iff $\lambda > 0$ belongs to the finite index set $PS(\tilde{\sigma}, \varphi) = PS_{\text{res}}(\tilde{\sigma}, \varphi) \cup PS_U(\tilde{\sigma}, \varphi)$ introduced in Definition 7.7. We define the Hilbert spaces

$$\mathcal{H}_{\text{res}}(\varphi) \ := \ \overset{\text{Hilbert}}{\underset{\{\sigma\in\hat{M} \mid p_\sigma(0)=0\}}{\bigoplus}} \underset{\lambda\in PS_{\text{res}}(\tilde{\sigma},\varphi)\backslash\{0\}}{\bigoplus} E_\Lambda(\tilde{\sigma}_\lambda, \varphi) \otimes I^{\sigma,\lambda}$$

$${}_0\mathcal{H}_U(\varphi) \ := \ \overset{\text{Hilbert}}{\underset{\{\sigma\in\hat{M} \mid p_\sigma(0)=0\}}{\bigoplus}} \underset{\lambda\in PS_U(\tilde{\sigma},\varphi)\backslash\{0\}}{\bigoplus} U_\Lambda(\tilde{\sigma}_\lambda, \varphi) \otimes I^{\sigma,\lambda} \ .$$

The sum $\mathcal{H}_{\text{res}}(\varphi) \oplus {}_0\mathcal{H}_U(\varphi)$ is the Hilbert space associated to the complementary series $\hat{G}_c$.

Now we consider the Hilbert space asssociated to the unitary principal series. First we discuss the contribution of $\pi^{\sigma,0}$. We decompose $\hat{M} = \cup_{i=1}^3 \hat{M}_i$ such that

1. $\sigma \in \hat{M}_1$ iff it is Weyl-invariant and $p_\sigma(0) = 0$ (i.e. $\pi^{\sigma,0}$ irreducible),

2. $\sigma \in \hat{M}_2$ iff it is Weyl-invariant and $p_\sigma(0) \neq 0$ (i.e. $\pi^{\sigma,0} = \pi^{\sigma,+} \oplus \pi^{\sigma,-}$), and

3. $\sigma \in \hat{M}_3$ iff is not Weyl-invariant.

We define

$$_1\mathcal{H}_U(\varphi) := \bigoplus_{\{\sigma \in \hat{M}_1 \,|\, 0 \in PS_U(\tilde{\sigma},\varphi)\}}^{\text{Hilbert}} U_\Lambda(\tilde{\sigma}_0, \varphi) \otimes L^2(\partial X, V(\sigma_0)) .$$

If $\sigma \in \hat{M}_2$, then $L^2(\partial X, V(\sigma_0)) = V_{\pi^{\sigma,+}} \oplus V_{\pi^{\sigma,+}}$, and we define

$$
\begin{aligned}
U_\Lambda^\pm(\tilde{\sigma}_0, \varphi) &:= \{f \in U_\Lambda(\tilde{\sigma}_0, \varphi) \,|\, \langle f, g \rangle = 0 \;\forall g \in V_{\pi^{\sigma,\mp},\infty}\} \\
&= \{f \in C^{-\infty}(\Lambda, V(\tilde{\sigma}_0, \varphi)) \,|\, J_0(f) = \pm f\} .
\end{aligned}
$$

We define

$$_2\mathcal{H}_U(\varphi) := \bigoplus_{\{\sigma \in \hat{M}_2 \,|\, 0 \in PS_U(\tilde{\sigma},\varphi)\}}^{\text{Hilbert}} (U_\Lambda^+(\tilde{\sigma}_0, \varphi) \otimes V_{\pi^{\sigma,+}}) \oplus (U_\Lambda^-(\tilde{\sigma}_0, \varphi) \otimes V_{\pi^{\sigma,-}}) .$$

Let $\hat{M}_4 \subset \hat{M}_3$ be a set of representatives of $\hat{M}_3/W(\mathfrak{g}, \mathfrak{a})$. We define

$$_3\mathcal{H}_U(\varphi) := \bigoplus_{\{\sigma \in \hat{M}_4 \,|\, 0 \in PS_U(\tilde{\sigma},\varphi)\}}^{\text{Hilbert}} U_\Lambda(\tilde{\sigma}_0, \varphi) \otimes L^2(\partial X, V(\sigma_0)) .$$

It seems to be natural to collect together all spaces connected with $U_\Lambda(\sigma_\lambda, \varphi)$, $\lambda \in PS_U(\sigma, \varphi)$, and to define

$$\mathcal{H}_U(\varphi) := \bigoplus_{i=0}^3 {}_i\mathcal{H}_U(\varphi) .$$

The main contribution of unitary principal series is the Hilbert space

$$\mathcal{H}_{ac}(\varphi) := \bigoplus_{\sigma \in \hat{M}}^{\text{Hilbert}} \mathcal{H}_{ac}^\sigma(\varphi) ,$$

where

$$\mathcal{H}_{ac}^\sigma(\varphi) := \int_{\mathfrak{a}_+^*}^\oplus L^2(B, V_B(\tilde{\sigma}_{i\lambda}, \varphi)) \hat{\otimes} L^2(\partial X, V(\sigma_{-i\lambda})) \, \frac{2\pi\omega_X}{\dim(V_\sigma)} p_\sigma(i\lambda) \, d\lambda .$$

For $\sigma \in \hat{M}$ we define the Wave packet transform

$$WP_\sigma : \mathcal{H}_0^{\tilde{\sigma}}(\varphi) \otimes L^2(\partial X, V(\sigma_{-i\lambda}))_K \to L^2(\Gamma\backslash G, \varphi)$$

in the following way (the space $\mathcal{H}_0^{\tilde{\sigma}}(\varphi)$ is defined just before Definition 10.6). Consider $T \in \text{Hom}_M(V_{\tilde{\sigma}}, V_{\tilde{\gamma}})$ for some $\gamma \in \hat{K}$, $v \in V_\gamma$, and $\phi \in \mathcal{H}_0^{\tilde{\sigma}}(\varphi)$. We take the element $v_T \in L^2(\partial X, V(\sigma_{-i\lambda}))_K$ (see Lemma 9.1) and form $\phi \otimes v_T$. Then we define

$$WP_\sigma(\phi \otimes v_T) = \langle E(\phi, T), v \rangle \ .$$

We employ the extension *ext* in order to identify the space $L^2(B, V_B(\tilde{\sigma}_{i\lambda}, \varphi))$ with a subspace of $^\Gamma(C^{-\infty}(\partial X, V(\tilde{\sigma}_{i\lambda})) \otimes V_\varphi)_{temp}$ which is our candidate of $N_{\pi^\sigma, -i\lambda}$. Then by Lemma 9.1 the wave packet transform WP_σ is related to the family of matrix coefficient maps $\{c_{\pi^\sigma, -i\lambda}\}_{\lambda \in \mathfrak{a}_+^*}$ by

$$WP_\sigma(\phi \otimes v_T) = \frac{2\pi\omega_X}{\dim(V_\sigma)} \int_{\mathfrak{a}_+^*} c_{\pi^\sigma, -i\lambda}(ext(\phi) \otimes v_T) p_\sigma(i\lambda) \, d\lambda \ . \tag{75}$$

Note that the elements of the form $\phi \otimes v_T$ span the dense subspace $\mathcal{H}_0^{\tilde{\sigma}}(\varphi) \otimes L^2(\partial X, V(\sigma_{-i\lambda}))_K$ of $\mathcal{H}_{ac}^\sigma(\varphi)$.

We are now able to state the Plancherel theorem for $L^2(\Gamma \backslash G, \varphi)$. Recall that in case $X = \mathbb{O}H^2$ we assume $\delta_\Gamma < 0$.

Theorem 11.1 *The direct sum of the matrix coefficient maps $c_\pi : {}^\Gamma(V_{\pi', -\infty} \otimes V_\varphi)_d \otimes V_{\pi, \infty} \to L^2(\Gamma \backslash G, \varphi)$, $\pi \in \hat{G}$, and the wave packet transforms WP_σ, $\sigma \in \hat{M}$, extends to a unitary equivalence of G-representations*

$$\mathcal{H}_{ac}(\varphi) \oplus \mathcal{H}_{cusp}(\varphi) \oplus \mathcal{H}_{\text{res}}(\varphi) \oplus \mathcal{H}_U(\varphi) \cong L^2(\Gamma \backslash G, \varphi) \ .$$

It gives rise to a corresponding decomposition

$$L^2(\Gamma \backslash G, \varphi) = L^2(\Gamma \backslash G, \varphi)_{ac} \oplus L^2(\Gamma \backslash G, \varphi)_d \ ,$$

where the discrete subspace

$$L^2(\Gamma \backslash G, \varphi)_d := L^2(\Gamma \backslash G, \varphi)_{cusp} \oplus L^2(\Gamma \backslash G, \varphi)_{\text{res}} \oplus L^2(\Gamma \backslash G, \varphi)_U$$

is the sum of the cuspidal, the residual, and the "stable" part.

$L^2(\Gamma \backslash G, \varphi)_d$ *is the sum of all irreducible subrepresentations of $L^2(\Gamma \backslash G, \varphi)$. $L^2(\Gamma \backslash G, \varphi)_{cusp}$ decomposes into discrete series representations of G, each discrete series representation of G occurs with infinite multiplicity. It is empty iff $X = \mathbb{R}H^n$, n odd. The remaining part of $L^2(\Gamma \backslash G, \varphi)_d$ consists of non-discrete series representations of G with real infinitesimal character occuring with finite multiplicity. If $\delta_\Gamma < 0$, then it is empty. If $\delta_\Gamma > 0$ and $\varphi = 1$, then it contains the representation I^{1, δ_Γ} with multiplicity one.*

$L^2(\Gamma \backslash G, \varphi)_{ac}$ *decomposes into a sum of direct integrals corresponding to the unitary principal series representations of G, each occuring with infinite multiplicity.*

The notions $L^2(\Gamma \backslash G, \varphi)_{cusp}$ and $L^2(\Gamma \backslash G, \varphi)_{\text{res}}$ are chosen in analogy with the case of groups Γ with finite covolume. Indeed, $L^2(\Gamma \backslash G, \varphi)_{\text{res}}$ is spanned by the residues of Eisenstein series.

However, the "cusp forms" forming the space $L^2(\Gamma\backslash G, \varphi)_{cusp}$ share the properties of the cusp forms in the sense of Harish-Chandra associated to the trivial group and not of those for groups with finite covolume. The appearance of $L^2(\Gamma\backslash G, \varphi)_U$ does not seem to have an analogue.

Proof of the theorem. It follows from Corollary 8.7, the determination of ${}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d$ in Section 9, and our definition of the scalar products on ${}^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d$ that the matrix coefficient maps induce a G-equivariant unitary map of $\mathcal{H}_{cusp}(\varphi) \oplus \mathcal{H}_{\mathrm{res}}(\varphi) \oplus \mathcal{H}_U(\varphi)$ onto the discrete subspace $L^2(\Gamma\backslash G, \varphi)_d$. Next we show that the wave packet transform WP_σ extends to a unitary embedding of $\mathcal{H}_{ac}^\sigma(\varphi)$ into $L^2(\Gamma\backslash G, \varphi)$. We compute using Proposition 10.8 and (64)

$$
\begin{aligned}
&\|WP_\sigma(\phi \otimes v_T)\|^2_{L^2(\Gamma\backslash G,\varphi)} \\
&= \ (\int_{\mathfrak{a}_+^*} \langle E(i\lambda, \phi_{i\lambda}, T), v\rangle \, p_\sigma(i\lambda)d\lambda \ , \ \langle E(\phi,T),v\rangle)_{L^2(\Gamma\backslash G,\varphi)} \\
&= \ \int_{\mathfrak{a}_+^*} \langle (E(i\lambda, \phi_{i\lambda}, T), v) \ , \ \langle E(\phi,T),v\rangle\rangle_{L^2(\Gamma\backslash G,\varphi)} \, p_\sigma(i\lambda)d\lambda \\
&= \ \int_{\mathfrak{a}_+^*} \int_{\Gamma\backslash G} \int_K \langle E(i\lambda, \phi_{i\lambda}, T)(gk), v\rangle \overline{\langle E(\phi,T)(gk),v\rangle} \, dk \, dg \, p_\sigma(i\lambda)d\lambda \\
&= \ \int_{\mathfrak{a}_+^*} \int_{\Gamma\backslash G} \int_K \langle \tilde\gamma(k^{-1})E(i\lambda, \phi_{i\lambda}, T)(g), v\rangle \overline{\langle \tilde\gamma(k^{-1})E(\phi,T)(g),v\rangle} \, dk \, dg \, p_\sigma(i\lambda)d\lambda \\
&= \ \frac{\|v\|^2}{\dim(V_\gamma)} \int_{\mathfrak{a}_+^*} \int_{\Gamma\backslash G} (E(i\lambda, \phi_{i\lambda}, T)(g), E(\phi,T)(g)) \, dg \, p_\sigma(i\lambda)d\lambda \\
&= \ \frac{2\pi\omega_X \|T\|^2 \|v\|^2}{\dim(V_\gamma)} \int_{\mathfrak{a}_+^*} (\phi_{i\lambda}, \phi_{i\lambda})_B \, p_\sigma(i\lambda)d\lambda \\
&= \ \frac{2\pi\omega_X \|v_T\|^2}{\dim(V_\sigma)} \int_{\mathfrak{a}_+^*} (\phi_{i\lambda}, \phi_{i\lambda})_B \, p_\sigma(i\lambda)d\lambda \\
&= \ \|\phi \otimes v_T\|^2_{\mathcal{H}_{ac}^\sigma(\varphi)} \ .
\end{aligned}
$$

By a similar computation we see that the images of the WP_σ, $\sigma \in \hat{M}$, are pairwise orthogonal. It follows from (75) that the extension of WP_σ to $\mathcal{H}_{ac}^\sigma(\varphi)$ is G-equivariant. As a subspace of the continuous subspace of $L^2(\Gamma\backslash G, \varphi)$ the image of WP_σ is also orthogonal to $L^2(\Gamma\backslash G, \varphi)_d$.

It remains to show that the image of $\mathcal{H}_{ac}(\varphi) \oplus \mathcal{H}_{cusp}(\varphi) \oplus \mathcal{H}_{\mathrm{res}}(\varphi) \oplus \mathcal{H}_U(\varphi)$ exhausts $L^2(\Gamma\backslash G, \varphi)$. Let $f \in L^2(\Gamma\backslash G, \varphi)$ be orthogonal to that image. We may and shall assume that it belongs to a K-isotypic component $L^2(\Gamma\backslash G, \varphi)(\gamma)$ for some $\gamma \in \hat{K}$. By Corollary 8.7 we can compute the scalar product of f with a Schwartz function $g \in \mathcal{C}(\Gamma\backslash G, \varphi)$ in the following way

$$
\begin{aligned}
(f,g) \ &= \ \int_{\hat{G}} (\mathcal{F}(f)(\pi), \mathcal{F}(g)(\pi))_{N_\pi \otimes V_\pi} \, d\kappa(\pi) \\
&= \ \int_{\hat{G}} (\mathcal{F}(f)(\pi), \mathcal{F}_\pi(g))_{N_\pi \otimes V_\pi} \, d\kappa(\pi)
\end{aligned}
$$

$$= \int_{\hat{G}} (c_\pi(\mathcal{F}(f)(\pi)), g)_{L^2(\Gamma\backslash G,\varphi)} \, d\kappa(\pi) \ . \tag{76}$$

Since f is orthogonal to $L^2(\Gamma\backslash G,\varphi)_d$ we have $\mathcal{F}(f)(\pi) = 0$ for all $\pi \in \hat{G}$ with $\kappa(\pi) \neq 0$, i.e., $^\Gamma(V_{\pi',-\infty} \otimes V_\varphi)_d \neq 0$. Thus, by the results of Section 9, it remains to show that $\mathcal{F}(f)(\pi) = 0$ for the unitary principal series representations $\pi^{\sigma,-i\lambda}$. Because of the equivalence $\pi^{\sigma,i\lambda} \cong \pi^{\sigma^w,-i\lambda}$ we can assume $\lambda \in \mathfrak{a}_+^*$. Clearly, $\mathcal{F}(f)(\pi^{\sigma,-i\lambda}) = 0$ if $[\gamma_{|M} : \sigma] = 0$.

By Corollary 8.7 and Lemma 9.4 we have

$$\mathcal{F}(f)(\pi^{\sigma,-i\lambda}) \in {}^\Gamma C^{-\infty}(\partial X, V(\tilde{\sigma}_{i\lambda},\varphi)) \otimes L^2(\partial X, V(\sigma_{-i\lambda}))(\gamma) \ .$$

For each $\sigma \in \hat{M}$, with $[\gamma_{|M} : \sigma] \neq 0$ let $\{T_i^\sigma\}_{i=1}^{\dim \, \mathrm{Hom}_M(V_{\tilde{\sigma}}, V_{\tilde{\gamma}})}$ be a basis of $\mathrm{Hom}_M(V_{\tilde{\sigma}}, V_{\tilde{\gamma}})$. Let $\{v^j\}_{j=1}^{\dim(V_\gamma)}$ be a basis of V_γ. We conclude that there are sections $\phi_{\sigma,i\lambda}^{ij} \in C^{-\infty}(B, V_B(\tilde{\sigma}_{i\lambda},\varphi))$, $i = 1, \ldots, \dim \, \mathrm{Hom}_M(V_{\tilde{\sigma}}, V_{\tilde{\gamma}})$, $j = 1, \ldots, \dim(V_\gamma)$, such that

$$\mathcal{F}(f)(\pi^{\sigma,-i\lambda}) = \sum_{i,j} ext(\phi_{\sigma,i\lambda}^{ij}) \otimes v_{T_i^\sigma}^j \ .$$

Using Lemma 9.1 we find

$$c_{\pi^{\sigma,-i\lambda}}(\mathcal{F}(f)(\pi^{\sigma,-i\lambda})) = \sum_{i,j} \langle E(i\lambda, \phi_{\sigma,i\lambda}^{ij}, T_i^\sigma), v^j \rangle \ .$$

We now evaluate the scalar product of f with some wave packet of the form $WP_{\sigma'}(\psi \otimes v_T)$, where ψ belongs to the space $\mathcal{H}_0^{\tilde{\sigma}'}(\varphi)$, $v \in V_\gamma$, and $T \in \mathrm{Hom}(V_{\tilde{\sigma}'}, V_{\tilde{\gamma}})$.

The map $\mathfrak{a}_+^* \ni \lambda \mapsto \pi^{\sigma,-i\lambda} \in \hat{G}$ identifies $\mathfrak{a}_+^*$ with an open subset of $\hat{G}$. Let $d\kappa(\sigma,\lambda)$ be the restriction of the Plancherel measure $d\kappa$ to this subset. Using (76), Proposition 10.8 we obtain

$$\begin{aligned}
0 &= (f, WP_{\sigma'}(\psi \otimes v_T))_{L^2(\Gamma\backslash G,\varphi)} \\
&= \int_{\hat{G}} (c_\pi(\mathcal{F}(f)(\pi)), WP_{\sigma'}(\psi \otimes v_T)) \, d\kappa(\pi) \\
&= \sum_{\sigma\in\hat{M},[\gamma_{|M}:\sigma]\neq 0} \sum_{i,j} \int_{\mathfrak{a}_+^*} (\langle E(i\lambda, \phi_{\sigma,i\lambda}^{ij}, T_i^\sigma), v^j \rangle, \langle E(\psi, T), v \rangle) \, d\kappa(\sigma,\lambda) \\
&= 2\pi\omega_X \sum_{i,j} (v_j, v)(T_i^{\sigma'}, T) \int_{\mathfrak{a}_+^*} (\phi_{\sigma',i\lambda}^{ij}, \psi_{i\lambda})_B \, d\kappa(\sigma',\lambda) \ .
\end{aligned}$$

Varying σ', T and v we see that for all i, j and σ'

$$\int_{\mathfrak{a}_+^*} (\phi_{\sigma',i\lambda}^{ij}, \psi_{i\lambda})_B \, d\kappa(\sigma',\lambda) = 0 \ .$$

Moreover, we are free to multiply ψ with a function $h \in C_c^\infty(\mathfrak{a}_+^*)$. Hence for all such h

$$\int_{\mathfrak{a}_+^*} h(\lambda)(\phi_{\sigma',i\lambda}^{ij}, \psi_{i\lambda})_B \, d\kappa(\sigma^l,\lambda) = 0 \ .$$

We conclude that $(\phi^{ij}_{\sigma',i\lambda}, \psi_{i\lambda})_B = 0$ for almost all $\lambda \pmod \kappa$. We choose a countable dense set $\{\psi_m\} \subset C^\infty(B, V_B(\tilde\sigma'_0, \varphi))$. Using a holomorphic trivialization of the family of bundles $\{V_B(\tilde\sigma'_\mu, \varphi)\}_{\mu \in \mathfrak{a}^*_\mathbb{C}}$ we extend these sections to families $\mathfrak{a}^*_\mathbb{C} \ni \mu \mapsto \psi_{m,\mu} \in C^\infty(B, V_B(\tilde\sigma'_\mu, \varphi))$. We form $B_m := \{\lambda | (\phi^{ij}_{\sigma',i\lambda}, \psi_{m,i\lambda}) \neq 0\} \subset \mathfrak{a}^*_+$. Then $\kappa(B_m) = 0$. Moreover let $U := \bigcup_m B_m$. Then $\kappa(U) = 0$, and we have $(\phi^{ij}_{\sigma',i\lambda}, \psi_{m,i\lambda}) = 0$ for all $\lambda \in \mathfrak{a}^*_+ \setminus U$ and all m. Thus $\phi^{ij}_{\sigma',i\lambda} = 0$ for $\lambda \in \mathfrak{a}^*_+ \setminus U$. Hence $\mathcal{F}(f)(\pi) = 0$ for almost all $\pi \pmod \kappa$. Therefore $f = 0$. This proves that the image of $\mathcal{H}_{ac}(\varphi) \oplus \mathcal{H}_{cusp}(\varphi) \oplus \mathcal{H}_{\mathrm{res}}(\varphi) \oplus \mathcal{H}_U(\varphi)$ is all of $L^2(\Gamma\backslash G, \varphi)$ and hence the theorem. $\square$

Let now γ be a finite-dimensional unitary representation of K. We want to draw consequences of Theorem 11.1 for the spectral decomposition of the algebra $\mathcal{Z}$ containing the Casimir operator Ω_G and acting by unbounded operators on $L^2(Y, V_Y(\gamma, \varphi))$. Let $D(G, \gamma)$ be the algebra of G-invariant differential operators on $V(\gamma)$. This algebra is a finitely generated module over $\mathcal{Z}$, where according to our convention the homomorphism of $\mathcal{Z}$ to $D(G, \gamma)$ is induced by the left regular representation of G. The algebra $D(G, \gamma)$ might be noncommutative, even if γ is irreducible (but then $X = \mathbb{H}H^n$ or $\mathbb{O}H^2$).

As before, we identify $C^\infty(Y, V_Y(\gamma, \varphi))$ with $[C^\infty(\Gamma\backslash G, \varphi) \otimes V_\gamma]^K$. This identification provides an isomorphism of $[\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{End}(V_\gamma)]^K$ with $D(G, \gamma)$, where the action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(\Gamma\backslash G, \varphi)$ is induced by the right regular representation. In particular, the decomposition of $L^2(\Gamma\backslash G, \varphi)$ with respect to G induces a decomposition of $L^2(Y, V_Y(\gamma, \varphi))$ with respect to $D(G, \gamma)$.

If π is any admissible representation of G, then the finite-dimensional space $\mathrm{Hom}_K(V_\pi, V_\gamma)$ has a natural structure of a $D(G, \gamma)$-module. In fact, the action of $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V_\gamma)$ on $\mathrm{Hom}(V_\pi, V_\gamma)$ induces an action of $D(G, \gamma) \cong [\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{End}(V_\gamma)]^K$ on $\mathrm{Hom}_K(V_\pi, V_\gamma)$. Note that if π is irreducible, then $\mathrm{Hom}_K(V_\pi, V_\gamma)$ is an irreducible $D(G, \gamma)$-module. Moreover, if γ is irreducible, then the functor $\pi \mapsto \mathrm{Hom}_K(V_\pi, V_\gamma)$ provides a one-to-one correspondence between equivalence classes of irreducible G-representations (strictly speaking of irreducible $(\mathfrak{g}, K)$-modules) containing γ and of irreducible $D(G, \gamma)$-modules (see [54], 3.5.4.). The induced action of $\mathcal{Z}$ on $\mathrm{Hom}_K(V_\pi, V_\gamma)$ is given by the infinitesimal character of π.

We define the Hilbert space and $D(G, \gamma)$-module

$$\mathcal{H}^\gamma_{cusp}(\varphi) := \bigoplus_{\pi \in \hat{G}_d} {}^\Gamma(V_{\pi, -\infty} \otimes V_\varphi)_d \otimes \mathrm{Hom}_K(V_\pi, V_\gamma) \ .$$

This sum is finite since every irreducible subrepresentation of γ only occurs in finitely many discrete series representations. There is a natural unitary $D(G, \gamma)$-equivariant map of $\mathcal{H}^\gamma_{cusp}(\varphi)$ into $L^2(\Gamma\backslash G, \varphi)$ which is given on each summand by

$${}^\Gamma(V_{\pi, -\infty} \otimes V_\varphi)_d \otimes \mathrm{Hom}_K(V_\pi, V_\gamma) \ni \phi \otimes t \mapsto t(\pi(g^{-1}) \otimes \mathrm{id})\phi) \in [L^2(\Gamma\backslash G, \varphi) \otimes V_\gamma]^K \ . \quad (77)$$

By the Frobenius reciprocity $\mathrm{Hom}_M(V_\sigma, V_\gamma) \cong \mathrm{Hom}_K(V_{\pi^{\sigma,\lambda}}, V_\gamma)$ the space $\mathrm{Hom}_M(V_\sigma, V_\gamma)$ carries a structure of a $D(G, \gamma)$-module depending on $\lambda \in \mathfrak{a}^*_\mathbb{C}$. We denote this module by $\mathrm{Hom}_M(V_\sigma, V_\gamma)_\lambda$. For $\mathrm{Re}(\lambda) > 0$ we consider $I^{\sigma,\lambda}$ as a submodule of $V_{\pi^{\sigma,\lambda}}$. We define the $D(G, \gamma)$-module $\overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda$ to be the quotient of $\mathrm{Hom}_M(V_\sigma, V_\gamma)_\lambda$ which corresponds to the quotient

$\mathrm{Hom}_K(I^{\sigma,\lambda}, V_\gamma)$ of $\mathrm{Hom}_K(V_{\pi^{\sigma,\lambda}}, V_\gamma)$ by Frobenius reciprocity. Using Lemma 5.5, 1., one can check that $\overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda$ is the quotient of $\mathrm{Hom}_M(V_\sigma, V_\gamma)_\lambda$ by the subspace which is annihilated by multiplication by $c_\gamma(\lambda)$.

Recall the definition of the scalar product $(T_1, T_2)\mathrm{id}_{V_\sigma} := T_2^* T_1$ on $\mathrm{Hom}_M(V_\sigma, V_\gamma)$. For $\lambda \in i\mathfrak{a}^*$ this scalar product is equal to the scalar product induced by Frobenius reciprocity $\mathrm{Hom}_M(V_\sigma, V_\gamma) \cong \mathrm{Hom}_K(L^2(\partial X, V(\sigma, \lambda)), V_\gamma)$ rescaled by $\dim(V_\sigma)^{-1}$. Note that if $\sigma \in \hat{M}_2$, then there is an orthogonal decomposition

$$\mathrm{Hom}_M(V_\sigma, V_\gamma)_0 = \mathrm{Hom}_M(V_\sigma, V_\gamma)_0^+ \oplus \mathrm{Hom}_M(V_\sigma, V_\gamma)_0^-$$

corresponding to the decomposition $\pi^{\sigma,0} = \pi^{\sigma,+} \oplus \pi^{\sigma,-}$.

If $\mathrm{Re}(\lambda) > 0$ and $\bar{\pi}^{\sigma,\lambda}$ is unitary, then we define the scalar product on $\overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda$ by

$$([T_1], [T_2]) := \frac{\dim(V_\sigma)}{c_\sigma(\lambda)}(c_\gamma(\lambda)T_1, T_2^w) \ ,$$

where $T_i \in \mathrm{Hom}_M(V_\sigma, V_\gamma)$ are representatives of $[T_i] \in \overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda$. Note that this scalar product coincides with the scalar product induced by $\overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda \cong \mathrm{Hom}_K(I^{\sigma,\lambda}, V_\gamma)$.

We define the Hilbert spaces and $D(G, \gamma)$-modules

$$\mathcal{H}_{\mathrm{res}}^\gamma(\varphi) := \bigoplus_{\{\sigma \in \hat{M} \,|\, [\gamma_{|M}:\sigma] \neq 0\,,\, p_\sigma(0)=0\}} \bigoplus_{\lambda \in PS_{\mathrm{res}}(\sigma,\varphi)\backslash\{0\}} E_\Lambda(\sigma_\lambda, \varphi) \otimes \overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda$$

$$_0\mathcal{H}_U^\gamma(\varphi) := \bigoplus_{\{\sigma \in \hat{M} \,|\, [\gamma_{|M}:\sigma] \neq 0\,,\, p_\sigma(0)=0\}} \bigoplus_{\lambda \in PS_U(\sigma,\varphi)\backslash\{0\}} U_\Lambda(\sigma_\lambda, \varphi) \otimes \overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda$$

$$_1\mathcal{H}_U^\gamma(\varphi) := \bigoplus_{\{\sigma \in \hat{M}_1 \,|\, [\gamma_{|M}:\sigma] \neq 0\,,\, 0 \in PS_U(\sigma,\varphi)\}} U_\Lambda(\sigma_0, \varphi) \otimes \mathrm{Hom}_M(V_\sigma, V_\gamma)_0$$

$$_2\mathcal{H}_U^\gamma(\varphi) := \bigoplus_{\{\sigma \in \hat{M}_2 \,|\, [\gamma_{|M}:\sigma] \neq 0\,,\, 0 \in PS_U(\sigma,\varphi)\}} U_\Lambda^+(\sigma_0, \varphi) \otimes \mathrm{Hom}_M(V_\sigma, V_\gamma)_0^+ \oplus U_\Lambda^-(\sigma_0, \varphi) \otimes \mathrm{Hom}_M(V_\sigma, V_\gamma)_0^-$$

$$_3\mathcal{H}_U^\gamma(\varphi) := \bigoplus_{\{\sigma \in \hat{M}_4 \,,\, [\gamma_{|M}:\sigma] \neq 0 \,|\, 0 \in PS_U(\sigma,\varphi)\}} U_\Lambda(\sigma_0, \varphi) \otimes \mathrm{Hom}_M(V_\sigma, V_\gamma)_0 \ .$$

All these sums are finite. We further define

$$\mathcal{H}_U^\gamma(\varphi) := \bigoplus_{i=0}^{3} {}_i\mathcal{H}_U^\gamma(\varphi) \ .$$

The matrix coefficient maps defined on $\mathcal{H}_{res}(\varphi) \oplus \mathcal{H}_U(\varphi)$ induces a unitary map of $D(G, \gamma)$-modules from $\mathcal{H}_{res}^\gamma(\varphi) \oplus \mathcal{H}_U^\gamma(\varphi)$ to $L^2(Y, V_Y(\gamma, \varphi))$ which is given by the Poisson transform on each summand (see Lemma 9.1). In particular, if $\lambda > 0$, then the Poisson transform factors over

$(E_\Lambda(\sigma_\lambda, \varphi) \oplus U_\Lambda(\sigma_\lambda, \varphi)) \otimes \overline{\mathrm{Hom}_M(V_\sigma, V_\gamma)}_\lambda$. By (62) the image under the Poisson transform of $\mathcal{H}^\gamma_{\mathrm{res}}(\varphi)$ consists exactly of the residues of Eisenstein series.

Last not least we introduce the absolute continuous part as the finite sum

$$\mathcal{H}^\gamma_{ac}(\varphi) := \bigoplus_{\sigma \in \hat{M}\,,\, [\gamma_{|M}:\sigma] \neq 0} \mathcal{H}^{\sigma;\gamma}_{ac}(\varphi) \,,$$

where $\mathcal{H}^{\sigma,\gamma}_{ac}$ is the direct integral of Hilbert spaces and $D(G,\gamma)$-modules

$$\mathcal{H}^{\sigma;\gamma}_{ac}(\varphi) := \int_{\mathfrak{a}^*_+}^\oplus L^2(B, V_B(\sigma, i\lambda)) \otimes \mathrm{Hom}_M(V_\sigma, V_\gamma)_{i\lambda} \, 2\pi \omega_X p_\sigma(i\lambda) \, d\lambda \,.$$

By Proposition 10.8 the wave packet transforms extend to isometric $D(G,\gamma)$-equivariant embeddings of $\mathcal{H}^\gamma_{ac}(\varphi)$ into $L^2(Y, V_Y(\gamma, \varphi))$. Note that $D(G,\gamma)$ acts on $\mathcal{H}^{\sigma;\gamma}_{ac}(\varphi)$ as an algebra of unbounded operators via multiplication by $\mathrm{End}(\mathrm{Hom}_M(V_\sigma, V_\gamma))$-valued polynomials on $\mathfrak{a}^*$.

Now the following theorem is an immediate consequence of Theorem 11.1.

Theorem 11.2 *The maps (77), the Poisson transforms and the wave packets of Eisenstein series combine to a unitary equivalence of $D(G,\gamma)$-representations*

$$\mathcal{H}^\gamma_{ac}(\varphi) \oplus \mathcal{H}^\gamma_{cusp}(\varphi) \oplus \mathcal{H}^\gamma_{\mathrm{res}}(\varphi) \oplus \mathcal{H}^\gamma_U(\varphi) \cong L^2(Y, V_Y(\gamma, \varphi)) \,.$$

It gives rise to a corresponding decomposition

$$L^2(Y, V_Y(\gamma, \varphi)) = L^2(Y, V_Y(\gamma, \varphi))_{ac} \oplus L^2(Y, V_Y(\gamma, \varphi))_d \,,$$

where the discrete subspace

$$L^2(Y, V_Y(\gamma, \varphi))_d := L^2(Y, V_Y(\gamma, \varphi))_{cusp} \oplus L^2(Y, V_Y(\gamma, \varphi))_{\mathrm{res}} \oplus L^2(Y, V_Y(\gamma, \varphi))_U$$

is this the sum of the cuspidal, the residual, and the "stable" part.

The algebra $\mathcal{Z}$ has pure point spectrum on $L^2(Y, V_Y(\gamma, \varphi))_d$, whereas its spectrum on $L^2(Y, V_Y(\gamma, \varphi))_{ac}$ consists of finitely many branches of absolute continuous spectrum of infinite multiplicity. $L^2(Y, V_Y(\gamma, \varphi))_{cusp}$ is a finite sum of infinite-dimensional eigenspaces. The remaining part of the discrete subspace is finite-dimensional. If $\delta_\Gamma < 0$, then it is empty. □

We conclude this paper by some comments on Theorem 11.2.

- It is clear from Corollary 8.7 that integration against Eisenstein series gives a map

$$\mathcal{C}(Y, V_Y(\gamma, \varphi)) \to \mathcal{H}^\gamma_{ac}(\varphi)$$

which we call Eisenstein-Fourier transform. It is a left-inverse of the wave packet transform, and it would be interesting to investigate its image.

- The decomposition of $L^2(Y, V_Y(\gamma, \varphi))$ given by Theorem 11.2 is finer than the spectral decomposition with respect to the Casimir operator. In particular, the Casimir operator can have eigenvalues embedded in the continuous spectrum. This is a kind of accident, because these embedded eigenvalues can be separated from the continuous spectrum by additional operators belonging to $D(G, \gamma)$ with one possible exception. Namely an eigenspace arising from $\oplus_{i=1}^3 i\mathcal{H}_U^\gamma(\varphi)$ contributes an eigenvalue lying at the bottom of one branch of the absolute contiuous spectrum of Ω_G.

- So far we have not presented any example where the space $L^2(Y, V_Y(\gamma, \varphi))_U$ is non-trivial. Here is one. It also sheds some light on the previous remark. Let $\Gamma \subset SL(2, \mathbb{R})$ be a cocompact Fuchsian group. Consider $\Gamma \subset SL(2, \mathbb{C})$ in the standard way. The limit set consists of the equator of the sphere S^2. If we interpret the equator as a 1-current, then it is not difficult to see that it defines an element of $U_\Lambda(\sigma_0, 1)$, where $\sigma = \sigma^2 \oplus \sigma^{2,w}$ is the representation of $M \cong U(1)$ corresponding to 1-forms. The corresponding square integrable 1-form on Y is harmonic, i.e., it contributes to the L^2-cohomology of Y (compare Mazzeo-Phillips [40].

- Even in the case $\delta_\Gamma > 0$ it can happen that $L^2(Y, V_Y(\gamma, \varphi))_d = 0$. For instance if Y is odd-dimensional, then Corollary 7.9 implies that the Dirac operator acting on spinors has pure absolute continuous spectrum $(-\infty, \infty)$.

- Using the meromorphic continuation of the Eisenstein series to all of $\mathfrak{a}_{\mathbb{C}}^*$ and Theorem 11.2 one can show that the resolvent kernel of $(\Omega - z)^{-1}$ on $L^2(Y, V_Y(\gamma, \varphi))$ extends meromorphically to a finite-sheeted branched cover of $\mathbb{C}$.

References

[1] S. Agmon. Spectral theory of Schrödinger operators on Euclidean and non-Euclidean spaces. *Comm. Pure Appl. Math*, 39 (1986), S3–S16.

[2] J. G. Arthur. Harmonic analysis of tempered distributions on semisimple Lie groups of real rank one. In *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, pages 13–100. Math. Surveys and Monographs 31, AMS, Providence, 1989.

[3] M. W. Baldoni Silva. The unitary dual of $Sp(n, 1)$, $n \geq 2$. *Duke Math. J.*, 48 (1981), 549–583.

[4] M. W. Baldoni Silva and D. Barbasch. The unitary spectrum for real rank one groups. *Invent. math.*, 72 (1983), 27–55.

[5] E. P. van den Ban. The principal series for a reductive symmetric space I. H-fixed distribution vectors. *Ann. scient. Éc. Norm. Sup.*, 21 (1988), 359–412.

[6] E. P. van den Ban and H. Schlichtkrull. Local boundary data of eigenfunctions on a Riemannain symmetric space. *Invent. math.*, 98 (1989), 639–657.

[7] E. P. van den Ban and H. Schlichtkrull. Fourier transforms on a semisimple symmetric space. *Invent. Math.*, 130 (1997), 517–574.

[8] J. N. Bernstein. On the support of Plancherel measure. *J. of Geom. and Phys.*, 5 (1988), 663–710.

[9] J. L. Brylinski and P. Delorme. Vecteurs distributions *H*-invariants pour les séries principales généralisées d'espaces symetriques reductifs et prolongement meromorphe d'intégrales d'Eisenstein. *Invent. math.*, 109 (1992), 619–664.

[10] U. Bunke and M. Olbrich. Gamma-cohomology and the Selberg zeta function. *J. reine angew. Math.*, 467 (1995), 199–219.

[11] U. Bunke and M. Olbrich. *Selberg Zeta and Theta Functions*. Akademie Verlag, 1995.

[12] U. Bunke and M. Olbrich. Resolutions of distribution globalizations of Harish-Chandra modules and cohomology. *J. reine angew. Math.*, 497 (1998), 47–81.

[13] U. Bunke and M. Olbrich. Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group. Preprint available at http://www.uni-math.gwdg.de/bunke/grkl.dvi, 1997 (To appear in *Ann. of Math.*).

[14] J. Carmona and P. Delorme. Base méromorphe de vecteurs distributions *H*-invariants pour les séries principales généralisées d'espaces symetriques reductifs. Equations fonctionelle. *J. Funct. Anal.*, 122 (1994), 152–221.

[15] W. Casselman. Canonical extensions of Harish-Chandra modules. *Canadian J. Math.*, 41 (1989), 385–438.

[16] Y. Colin de Verdiére. Théorie spectrale et surfaces de Riemann d'aire infinie. *Asterisque*, 132 (1985), 259–275.

[17] D. H. Collingwood. *Representations of Rank One Lie Groups*. Pitman, Boston, Research Notes in Math. 137, 1985.

[18] K. Corlette. Hausdorff dimensions of limit sets. *Invent. math.*, 102 (1990), 521–542.

[19] P. Delorme. Formule de Plancherel pour les espaces symétriques réductifs. *Ann. of Math.*, 147 (1998), 417–452.

[20] J. Elstrodt. Die Resolvente zum Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene III. *Math. Annalen*, 208 (1975), 99–132.

[21] C. L. Epstein and R. B. Melrose. Shrinking tubes and the $\bar{\partial}$-Neumann problem. Preprint, 1991.

[22] C. L. Epstein, R. B. Melrose, and G. A. Mendoza. Resolvent of the Laplacian on strictly pseudoconvex domains. *Acta Math.*, 167 (1991), 1–106.

[23] J. D. Fay. Fourier coefficients of the resolvent for a Fuchsian group. *J. reine angew. Math.*, 293 (1977), 143–203.

[24] R. Froese, P. Hislop, and P. Perry. A Mourre estimate and related bounds for hyperbolic manifolds with cusps of non-maximal rank. *J. Funct. Anal.*, 98 (1991), 292–310.

[25] Harish-Chandra. Two theorems on semisimple Lie groups. *Ann. of Math.*, 83 (1966), 74–128.

[26] Harish-Chandra. Harmonic analysis on real reductive groups III. The Maass-Selberg relations and the Plancherel formula. *Ann. of Math.*, 104 (1976), 117–201.

[27] H. Hecht and J. L. Taylor. A remark on Casselman's comparison theorem. In *Geometry and Representation Theory of Real and p-Adic Groups*, pages 139–146. Progr. in Math. 158, Birkhäuser Boston, 1998.

[28] S. Helgason. *Groups and Geometric Analysis.* Academic Press, Inc., 1984.

[29] S. Helgason. *Geometric Analysis on Symmmetric Spaces.* Math. Surveys 39, A.M.S. Providence, R. I., 1994.

[30] T. Hirai. The Plancherel formula of the Lorentz group of n-th order. *Proc. Japan Acad.*, 42 (1966), 323–326.

[31] A. W. Knapp. *Representation Theory of Semisimple Lie Groups. An Overview Based on Examples.* Princeton University Press, 1986.

[32] A. W. Knapp and E. M. Stein. Intertwining operators for semisimple Lie groups. *Ann. of Math.*, 93 (1971), 489–578.

[33] P. D. Lax and R. S. Phillips. The asymptotic distribution of lattice points in Euclidian and non-Euclidian spaces. *J. Funct. Anal.*, 46 (1982), 280–350.

[34] P. D. Lax and R. S. Phillips. Translation representations for automorphic solutions of the wave equation in non-Euclidian spaces I/II. *Comm. Pure Appl. Math.*, 37 (1984), 303–328/779–813.

[35] P. D. Lax and R. S. Phillips. Translation representations for automorphic solutions of the wave equation in non-Euclidian spaces III. *Comm. Pure Appl. Math.*, 38 (1985), 197-208.

[36] N. Mandouvalos. The theory of Eisenstein series for Kleinian groups. *Contemp. Math*, 53 (1986), 358–370.

[37] N. Mandouvalos. Spectral theory and Eisenstein series for Kleinian groups. *Proc. London Math. Soc.*, 57 (1988), 209–238.

[38] N. Mandouvalos. *Scattering operator, Eisenstein series, inner product formula and "Maass-Selberg" relations for Kleinian groups.* Memoirs of the Am. Math. Soc. 78, 1989.

[39] R. Mazzeo and R. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75 (1987), 260–310.

[40] R. Mazzeo and R. Phillips. Hodge theory on hyperbolic manifolds. *Duke Math. J.*, 60 (1990), 509–559.

[41] K. Okamoto. On the Plancherel formulas for some types of simple Lie groups. *Osaka J. Math.*, 2 (1965), 247–282.

[42] M. Olbrich. *Die Poisson-Transformation für homogene Vektorbündel.* PhD thesis, Humboldt-Universität zu Berlin, 1995.

[43] S. J. Patterson. The Selberg zeta-function of a Kleinian group. In *Number Theory, Trace Formulas and Discrete Groups : Symposium in Honor of Atle Selberg, Oslo, Norway, 1987*, pages 409–441.

[44] S. J. Patterson. The Laplace operator on a Riemann surface I. *Comp. Math.*, 31 (1975), 83–107.

[45] S. J. Patterson. The Laplace operator on a Riemann surface II. *Comp. Math.*, 32 (1976), 71–112.

[46] S. J. Patterson. The Laplace operator on a Riemann surface III. *Comp. Math.*, 33 (1976), 227–259.

[47] S. J. Patterson. The limit set of a Fuchsian group. *Acta Math.*, 136 (1976), 241–273.

[48] P. A. Perry. The Laplace operator on a hyperbolic manifold I. Spectral and scattering theory. *J. Funct. Anal.*, 75 (1987), 161–187.

[49] P. A. Perry. The Laplace operator on a hyperbolic manifold II. Eisenstein series and the scattering matrix. *J. Reine angew. Math.*, 398 (1989), 67–91.

[50] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV. Analysis of Operators.* Academic Press, INC., 1978.

[51] H. Schlichtkrull. *Hyperfunctions and Harmonic Analysis on Symmetric Spaces.* Birkhäuser, Boston, 1984.

[52] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *I.H.E.S. Publ. Math.*, 50 (1979), 171–209.

[53] H. van der Ven. Vector valued Poisson transforms on Riemannian symmetric spaces of rank one. *J. Funct. Anal.*, 119 (1994), 358-400.

[54] N. R. Wallach. *Real Reductive Groups.* Academic Press, 1988.

[55] N. R. Wallach. *Real Reductive Groups II.* Academic Press, 1992.

[56] G. Warner. *Harmonic Analysis on Semi-Simple Lie Groups I.* Springer-Verlag, New York, 1972.

[57] An Yang. Poisson transforms on vector bundles. Preprint, Boston, 1994.